Ann. Rep. Asahikawa
Med. Coll.
1988. Vol.9, 1 ~ 14

SPECIAL SUBSPACES IN A FINSLER SPACE

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Introduction. In the previous paper [6][1], we developed a thoery of subspaces in a Finsler space through three kinds of connections (Matsumoto connections, *TMD*-connections and *TM*-connections). As for special subspaces, however, we could not make a full discussion of them there.

The principal purpose of the present paper is to make up for the above insufficiency. The terminologies and notations refer to papers [2] ~ [7] unless otherwise stated.

§1. Preliminaries. Let M_n be an n-dimensional Finsler space with a fundamental function $L(x^i, y^i)$, and be endowed with a Matsumoto connection $M\Gamma = (\Gamma_j{}^i{}_k, \Gamma^i{}_k, \tilde{C}_j{}^i{}_k)$. This connection is defined as follows ([4], [5]): The v-connection is given by a (-1) p-homogeneous[2] tensor $\tilde{C}_j{}^i{}_k$. The non-linear connection and the h-connection are given by

$$(1.1)\qquad \Gamma^i{}_k = G^i{}_k + T^i{}_k,$$

$$(1.2)\qquad \Gamma_j{}^i{}_k = \Gamma^i{}_{k\|j} + Q_j{}^i{}_k = G_j{}^i{}_k + T_j{}^i{}_k + Q_j{}^i{}_k,$$

where the symbol $\|j$ indicates the partial differentiation by y^j, $G^i{}_k$ and $G_j{}^i{}_k$ $(= G^i{}_{k\|j})$ are the non-linear connection and the h-connection of Berwald, $T^i{}_k$ and $Q_j{}^i{}_k$ are (1) p- and (0) p-homogeneous tensors respectively and $T_j{}^i{}_k = T^i{}_{k\|j}$.

An $M\Gamma$ is called a TMD(resp. $TMD(0)$)-connection and denoted by $TMD\Gamma$ (resp. $TMD\Gamma_0$) if the tensors $T^i{}_k$, $D^i{}_k$, $Q_j{}^i{}_k$ and $\tilde{C}_j{}^i{}_k$ are given as follows:

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1) Numbers in brackets refer to the references at the end of the paper.

2) "(r) p-homogeneous" means "positively homogeneous of degree r in y^i".

(1.3) $T^o{}_k = T^i{}_o = 0,\ \tilde{C}^i_{jk} = C^i_{jk}$ (resp. $\tilde{C}^i_{jk} = 0$),

(1.4) $D_{jk} + Q_{jok} = 0,\ D_{jk} = g_{js}D^s{}_k,\ D^i{}_k = Q^i_{ok}$,

where g_{is} and C^i_{jk} are the metric tensor and the C-tensor on M_n respectively, the upper or lower index o indicates contraction by y_i or y^i and $Q_{jik} = g_{is} Q^s_{jk}$.

A $TMD\Gamma$ (resp. $TMD\Gamma_o$) is called a TM (resp. $TM(O)$)-connection and denoted by $TM\Gamma$ (resp. $TM\Gamma_o$) if the tensor Q^i_{jk} satisfies

(1.5) $Q^i_{ok} = Q^o_{jk} = 0.$

Note 1.1. An $M\Gamma$ is a quite general connection with no metrical property. $TMD\Gamma$ (resp. $TMD\Gamma_o$) satisfies the following axioms: (F1) metrical ($L_{|k} = 0$). (F3) v-metrical and v-symmeyric ($\tilde{C}^i_{jk} = C^i_{jk}$) (resp. (F3)$_1$ v-natural ($\tilde{C}^i_{jk} = 0$)). (F4) Dy-reciprocal ($y^iDg_{ij} = 0$). (F5) a geo-path connection (paths with respect to this connection are always geodesics of M_n). A $TM\Gamma$ (resp. $TM\Gamma_o$) further satisfies the axiom: (F2) dft-free ($D^i{}_k = 0$). This connection is a slight generalization of the following connections: Cartan connection $C\Gamma$, Hashiguchi one $H\Gamma$, Rund one $R\Gamma$ and Berwald one $B\Gamma$ etc.

Let M_m be an m-dimensional subspace of M_n represented parametrically by the equation

(1.6) $x^i = x^i(u^a)\ (i = 1, 2, \cdots\cdots. n; a = 1, 2, \cdots\cdots. m).$

where we suppose that variables u^a form a coordinate sysyem of M_m and the matrix with components $B^i{}_a\ (= \partial x^i / \partial u^a)$ is of rank m.

If we denote the components of a vector y^i tangent to a curve in M_m by y^a [3] in terms of u^a-system, then we have

(1.7) $y^i = B^i{}_a y^a,\quad y^i{}_{\|a} := \partial y^i / \partial y^a = B^i{}_a.$

The induced fundamental function $\bar{L}(u^a, y^a)$ and the metric tensor $g_{\beta\gamma}(u^a, y^a)$ on M_m are given by

(1.8) $\bar{L} = L(x^i(u^a), B^i{}_a y^a),\ g_{\beta\gamma} = g_{jk}B^{jk}_{\beta\gamma} := g_{jk}B^j{}_\beta B^k{}_\gamma.$

We choose n-m unit normal vectors N^i_a ($a = m+1, \cdots\cdot, n$) at each point (u^a) of M_m such that

3) If no confusion occurs, then we shall use y^a in stead of the usual notation v^a.

(1.9) $$g_{ij}N^i_a N^j_b = \delta_{ab},\ B^i_{\ \alpha}N^b_i = 0,\ N^b_i := g_{ij}N^j_b.$$

If we put $B^\alpha_i = g_{ij}B^j_{\ \beta}g^{\alpha\beta}$, where $g^{\alpha\beta}$ is the reciprocal tensor of $g_{\alpha\beta}$, then the inverse matrix of $(B^i_{\ \alpha},\ N^i_a)$ is given by $(B^\alpha_i,\ N^a_i)$. In this case, the following relations hold:

(1.10) $$B^\alpha_{i\,\|\gamma} = C_b{}^\alpha{}_\gamma N^b_i,\quad C_b{}^\alpha{}_\gamma := C^i_{jk}N^j_b\, B^\alpha_i B^k{}_\gamma.$$

(1.11) $$N^i_{a\,\|\gamma} = -2C_a{}^\beta{}_\gamma B^i{}_\beta - \lambda_a{}^b{}_\gamma N^i_b,\quad \lambda_a{}^b{}_\gamma := N^b_j N^j_{a\,\|\gamma}.$$

(1.12) $$\lambda_b{}^a{}_\gamma + \lambda_a{}^b{}_\gamma = 2C_b{}^a{}_\gamma = 2C_a{}^b{}_\gamma,\quad C_b{}^a{}_\gamma := C^i_{jk}N^a_i N^j_b\, B^k{}_\gamma.$$

The induced Matsumoto connection $IM\Gamma = (\Gamma_\beta{}^\alpha{}_\gamma,\ \Gamma^\alpha{}_\gamma,\ \widetilde{C}_\beta{}^\alpha{}_\gamma)$ on M_m is defined as follows [6]:

(1.13) $$\Gamma_\beta{}^\alpha{}_\gamma = B^\alpha_i(B^i_{\beta\gamma} + \Gamma^i_{jk}B^{j\,k}_{\beta\,\gamma}) + \widetilde{C}_\beta{}^\alpha{}_a H^a_\gamma\ ,$$

(1.14) $$\Gamma^\alpha{}_\gamma = B^\alpha_i(B^i_{0\gamma} + \Gamma^i{}_k B^k{}_\gamma),\ \widetilde{C}_\beta{}^\alpha{}_\gamma = \widetilde{C}^i_{jk}B^\alpha_i B^{j\,k}_{\beta\,\gamma},$$

where we put

(1.15) $$B^i_{\beta\gamma} = \partial B^i{}_\gamma / \partial u^\beta,\ B^i_{0\gamma} = y^\beta B^i_{\beta\gamma},\ \widetilde{C}_\beta{}^\alpha{}_b = \widetilde{C}^i_{jk}B^\alpha_i B^j{}_\beta N^k_b\ ,$$

(1.16) $$H^a{}_\gamma = N^a_i(B^i_{0\gamma} + \Gamma^i{}_k B^k{}_\gamma).$$

The normal curvature vector in a direction N^i_a is given by (1.16), while the second fundamental tensor in the same direction is given by

(1.17) $$H_\beta{}^a{}_\gamma = N^a_i(B^i_{\beta\gamma} + \Gamma^i_{jk}B^{j\,k}_{\beta\,\gamma}) + \widetilde{C}_\beta{}^a{}_b H^b_\gamma\ ,$$

where $\widetilde{C}_\beta{}^a{}_b = \widetilde{C}^i_{jk}B^j{}_\beta N^a_i N^k_b$.

Let $\widetilde{R}_{\alpha\delta\beta\gamma}$ be the h-curvature tensor with respect to $IM\Gamma$. If we contract this tensor by $y^\alpha y^\beta$, then we have

$$\widetilde{R}_{0\delta 0\gamma} = \widetilde{R}_{0i0h}B^{i\,h}_{\delta\,\gamma} + \widetilde{S}_{0ikh}B^i_\delta\, N^k_a\, N^h_b\, H^a_0\, H^b_\gamma$$

(1.18) $$+ B^i{}_\delta N^h_a\,(\widetilde{P}_{0i0h}H^a_\gamma - \widetilde{P}_{0ikh}B^k{}_\gamma H^a_0) + H_0{}^a{}_0(g_{jk|\gamma}B^j{}_\delta N^k_a + \delta_{ab}H_\delta{}^b{}_\gamma)$$

$$- H_0{}^a{}_\gamma(g_{jk|\beta}\, y^\beta\, B^j{}_\delta\, N^k_a + \delta_{ab}H_\delta{}^b{}_0),$$

where $\widetilde{R}_{0i0h} = \widetilde{R}_{jikh}y^j y^k$, $\widetilde{S}_{0ikh} = \widetilde{S}_{jikh}y^j$, $\widetilde{P}_{0i0h} = \widetilde{P}_{jikh}y^j y^k$ and $\widetilde{R}_{jikh}$, $\widetilde{S}_{jikh}$, $\widetilde{P}_{jikh}$ are the h-, v-, hv- curvature tensors respectively.

§ 2. Totally n-parallel subspaces. In the previous paper [6], we investigated the following subspaces:

(a) a totally geodesic subspace M_m of M_n, in which each geodesic of M_m is always a geodesic of M_n.

(b) a totally auto-parallel subspace M_m of M_n, in which each path of M_m with respect to $IM\Gamma$ is always a path of M_n with respect to $M\Gamma$.

(c) a totally h-auto-parallel subspace M_m of M_n, in which each h-path of M_m with respect to $IM\Gamma$ is always an h-path of M_n with respect to $M\Gamma$.

In this section, we shall consider another parallel subspace.

We shall say that M_m is a *totally n-parallel subspace* (or simply *totally n-parallel*) *with respect to IM*Γ if each normal vector N_a^i is parallel along any curve in M_m with respect to $IM\Gamma$.

Note 2. 1. Subspaces (b) and (c) correspond to a hyperplane of the first kind and of the second kind respectively in the theory of hypersurfaces ([1], [2]), while the above new subspace corresponds to that of the third kind.

The absolute differential of N_a^i is given by

$$DN_a^i = N_{a\ |\gamma}^i du^\gamma + N_a^i|_\gamma Dy^\gamma, \tag{2. 1}$$

where

$$N_{a\ |\gamma}^i = (N_j^b N_{a\ |\gamma}^j) N_b^i - g^{\beta\varepsilon}(g_{jk|\gamma} B^j_\beta N_a^k + \delta_{ab} H_\beta{}^b{}_\gamma) B^i{}_\varepsilon, \tag{2. 2}$$

$$N_a^i|_\gamma = (N_j^b N_a^j|_\gamma) N_b^i - g^{\beta\varepsilon}(g_{jk}|_\gamma B^j_\beta N_a^k + \delta_{ab} \widetilde{C}_\beta{}^b{}_\gamma) B^i{}_\varepsilon. \tag{2. 3}$$

Let M_m be totally n-parallel. Then from (2. 1) $\sim$ (2. 3) we obtain

$$g_{jk|\gamma} B^j_\beta N_a^k + \delta_{ab} H_\beta{}^b{}_\gamma = 0, \tag{2. 4}$$

$$g_{jk}|_\gamma B^j_\beta N_a^k + \delta_{ab} \widetilde{C}_\beta{}^b{}_\gamma = 0, \tag{2. 5}$$

$$N_j^b N_{a\ |\gamma}^j = 0, \tag{2. 6}$$

$$N_j^b N_a^j|_\gamma = 0. \tag{2. 7}$$

If we put $\widetilde{C}_{k\ \gamma}^{\ i} = \widetilde{C}_{k\ h}^{\ i} B^h{}_\gamma$, then we have $N_a^j|_\gamma = N_{a\ \|\gamma}^j + \widetilde{C}_{k\ \gamma}^{\ j} N_a^k$. Therefore it follows from (1. 11) that $N_j^b N_a^j|_\gamma = -\lambda_{a\ \gamma}^{\ b} + \widetilde{C}_a{}^b{}_\gamma$, where $\widetilde{C}_a{}^b{}_\gamma = \widetilde{C}_{k\ \gamma}^{\ i} N^k{}_a\ N_i^b$.

Consequently the condition (2. 7) is equivalent to

(2.8) $\lambda_{a\ \gamma}^{\ b} = \tilde{C}_{a\ \gamma}^{\ b}.$

Differentiating the first in (1.9) h-covariantly by u^γ, we have

$$g_{ij|\gamma} N_a^i N_b^j + N_j^a N_{b\ |\gamma}^j + N_j^b N_{a\ |\gamma}^j = 0,$$

which shows that the condition (2.6) is equivalent to

(2.9) $g_{ij|\gamma} N_a^i N_b^j = 0,\ N_j^a N_{b\ |\gamma}^j = N_j^b N_{a\ |\gamma}^j.$

Consequently we can state

Theorem 2.1. *A subspace M_m of M_n is totally n-parallel with respect to $IM\Gamma$ if and only if equations* (2.4), (2.5), (2.8) *and* (2.9) *hold.*

The projection factors B^i_α are independent of a direction y^α, while the reciprocal ones B_i^α are dependent on it. Then it follows from (1.10) that $B_{i\ \|\gamma}^{\alpha} = 0$ if and only if the following equation holds:

(2.10) $C_{b\ \gamma}^{\ a} (= C_{\gamma\ b}^{\ a} = C_{a\ \gamma}^{\ b} = C_{a\gamma b}) = 0.$

We shall say that M_m is *projection factor-direction-free* or simply *pfd-free* if the equation (2.10) holds on M_m.

Note 2.2. It is known [6] that if M_m is pfd-free, then the induced connection $IM\Gamma$ is the intrinsic one on M_m.

Suppose that the $M\Gamma$ is an h-metrical $TM\Gamma$ (or $TMD\Gamma$). Then we have

(2.11)
$$\tilde{C}_{\beta\ \gamma}^{\ b} = C_{\beta\ \gamma}^{\ b},\ g_{ij}|_\gamma = g_{ij}|_h B^h_{\ \gamma} = 0,$$
$$g_{ij|\gamma} = g_{ij|h} B^h_{\ \gamma} + g_{ij}|_h N_a^h H_\gamma^a = 0.$$

Therefore, from (2.4), (2.5), (2.8), (2.9) and (2.11) we obtain

(2.12) $C_{\beta\ \gamma}^{\ a} = 0,\quad H_{\beta\ \gamma}^{\ a} = 0,$

(2.13) $\lambda_{b\gamma}^{a} = C_{b\ \gamma}^{\ a},\quad N_j^a N_{b\ |\gamma}^j = N^b_{\ j} N_{a\ |\gamma}^j.$

Consequently we can state

Corollary 2.1.1. *Let the connection $M\Gamma$* in *consideration be an h-metrical $TM\Gamma$ (or $TMD\Gamma$). Then a subspace M_m is totally n-parallel with respect to $IM\Gamma$ if and only if M_m is pfd-free, each second fundamental tensor vanishes and the relation*

(2. 13) *holds.*

If the $M\Gamma$ is an h-metrical $TM\Gamma_o$(or $TMD\Gamma_o$), then we obtain

$$(2.14)\qquad \tilde{C}_\beta{}^a{}_\gamma = 0,\ g_{ij}|_\gamma = 2C_{ij\gamma},\ g_{ij|\gamma} = 2C_{ija}H^a_\gamma\ ,$$

where $C_{ij\gamma} = C_{ijk}B^k{}_\gamma$ and $C_{ija} = C_{ijk}N^k_a$. Therefore, by virtue of (1. 12), (2. 14) and Theorem 2. 1 we first obtain (2. 12) and then

$$(2.15)\qquad \lambda^a_{b\gamma} = C_b{}^a{}_\gamma (= C_{ab\gamma}) = 0,$$

$$(2.16)\qquad C_{abc}H^c_\gamma = 0,\ N^a_j N^j_{b|\gamma} = N^b_j N^j_{a|\gamma},\ C_{abc} := C_{ijk}N^i_a N^j_b N^k_c.$$

Consequently we can state

Corollary 2. 1. 2. *Let the connection $M\Gamma$ in consideration be an h-metrical $TM\Gamma_o$ (or $TMD\Gamma_o$). Then a subspace M_m is totally n-parallel with respect to $IM\Gamma$ if and only if M_m is pfd-free, eaeh second fundamental tensor vanishes and the relations* (2. 15) *and* (2. 16) *hold.*

In the following, we shall call a $TM\Gamma$ or a $TMD\Gamma$ (resp. a $TM\Gamma_o$ or a $TMD\Gamma_o$) a *T-connection* (resp. a *T(0)-connection*) generically and denote it by $T\Gamma$ (resp. $T\Gamma_o$).

For the induced T(or $T(0)$)-connection $IT\Gamma$(or $IT\Gamma_o$), we have

$$(2.17)\qquad H_o{}^a{}_\gamma := y^\beta H_\beta{}^a{}_\gamma = H^a_\gamma + D^a{}_\gamma,\ D^a{}_\gamma := D^i{}_k N^a_i B^k{}_\gamma.$$

We shall say that the induced connection $IT\Gamma$ (or $IT\Gamma_o$) *satisfies the D-condition* if each $D^a{}_\gamma$ vanishes. Then we can state

Lemma 2. 1. *The following induced connectons satisfy the D-condition*:

(1) *All the induced TM(or TM(0))-connections.*

(2) *The induced AMD(or AMD(0))-connections.*

(3) *The induced MD, AMBD, AMCD-connectios (or respective corresponding (0)-connections).*

Proof. For (1), from (1. 5) we have $D^i{}_k = Q_o{}^i{}_k = 0$ and hence $D^a{}_\gamma = 0$. Next, an AMD(or $AMD(0)$)-connection is an h-metrical TMD (or $TMD(0)$) -connection defined by

$$(2.18)\qquad \Gamma^i{}_k = G^i{}_k + fLh^i{}_k,\ \Gamma^i_{jk} = \Gamma^{*i}_{jk} - fLC^i_{jk},\ D^i{}_k = -fLh^i{}_k,$$

where f is a (0) p-homogeneous scalar, Γ^{*i}_{jk} is the h-connection of $C\Gamma$ and $h^i{}_k$ is the angular metric tensor. In this case, we have $D^a{}_\gamma = 0$ because of (1.9). Lastly, an MD(or $MD(0)$)-connection is also an h-metrical TMD (or $TMD(0)$) -connection defined by

$$\Gamma^i{}_k = G^i{}_k,\ \Gamma^i_{jk} = \Gamma^{*}{}^i_{jk} + f(l_j\delta^i{}_k - l^i g_{jk}),\ D^i{}_k = fLh^i{}_k, \tag{2.19}$$

which implies $D^a{}_\gamma = 0$. Successively, an $AMBD$ (or $AMBD(0)$) -connection and an $AMCD$(or $AMCD(0)$)-connection are both h-symmetric TMD(or $TMD(0)$)-connections defined as follows: $\Gamma^i{}_k = G^i{}_k,\ \Gamma^i_{jk} = G^i_{jk} + Q^i_{jk}$,

$$Q^i_{jk} = f(l_j h^i{}_k + l_k h^i{}_j - l^i h_{jk}) - P^i_{jk} \qquad (AMBD\Gamma(\text{or } \Gamma_0)), \tag{2.20}$$

$$Q^i_{jk} = f(l_j h^i{}_k + l_k h^i{}_j - l^i h_{jk}) \qquad (AMCD\Gamma(\text{or } \Gamma_0)), \tag{2.21}$$

where P^i_{jk} is the hv-torsion tensor of $C\Gamma$. Contracting (2.20) and (2.21) by y^j, we have $D^i{}_k = fLh^i{}_k$ and hence $D^a{}_\gamma = 0$. Q. E. D.

Note 2.3. In Corollaries 2.1.1 and 2.1.2, M_m is also a totally geodesic subspace if the $IM\Gamma$ satisfies the D-condition.

Note 2.4. Practical examples for Corollaries 2.1.1 and 2.1.2 are as follows ([2], [5], [6]):

$TM\Gamma$: $C\Gamma$, $IS\Gamma$, $AMR\Gamma$ ⋯　　$TMD\Gamma$: $AMD\Gamma$, $CD\Gamma$, $MD\Gamma$ ⋯

$TM\Gamma_0$: $R\Gamma$, $IS\Gamma_0$, $AMR\Gamma_0$ ⋯　　$TMD\Gamma_0$: $AMD\Gamma_0$, $RD\Gamma$, $MD\Gamma_0$ ⋯

Since $N^a_{i\|\beta} = \lambda^a_{b\gamma} N^b_i$, from (1.16) and (1.17) we have

$$H^a_{\gamma\|\beta} = (\lambda^a_{b\beta} - \widetilde{C}_\beta{}^a{}_b)\, H^b_\gamma + H_\beta{}^a{}_\gamma - Q_\beta{}^a{}_\gamma,\ Q_\beta{}^a{}_\gamma := Q^i_{jk} N^a_i B^{jk}_{\beta\gamma}. \tag{2.22}$$

If $Q_\beta{}^a{}_\gamma = 0$, then we have $y^\beta Q_\beta{}^a{}_\gamma = D^a{}_\gamma = 0$. Consequently, by virtue of (2.17) and (2.22) we can state

Lemma 2.2. *Suppose that the connection $M\Gamma$ in consideration is a $T\Gamma$(or $T\Gamma_0$) and the induced connection $IM\Gamma$ satisfies $Q_\beta{}^a{}_\gamma = 0$. Then each $H_\beta{}^a{}_\gamma$ vanishes if and only if each H^a_γ vanishes, that is, M_m is totally geodesic.*

Further we can state

Lemma 2.3. *The following facts hold* :

(a) *If M_m is pfd-free, then a relation $C_{\beta ah} T^h{}_\gamma = C_{\beta ab} T^b{}_\gamma$ holds,*

(b) *A condition $C_{ab\gamma} = 0$ implies $C_{abh} T^h{}_\gamma = C_{abc} T^c{}_\gamma$,* *where* $C_{\beta ah} = C_{ijh} B^i_\beta N^j_a$, $C_{\beta ab} = C_{ijk} B^i_\beta N^j_a N^k_b$, $C_{abh} = C_{ijh} N^i_a N^j_b$, $T^h_\gamma = T^h_k B^k_\gamma$ *and* $T^a_\gamma = T^i_k N^a_i B^k_\gamma$.

Proof. Since $B^j_\varepsilon B^\varepsilon_h = \delta^j_h - N^j_b N^b_h$, we have

$$C_{\beta a\varepsilon} T^\varepsilon_\gamma = (C_{\beta aj} B^j_\varepsilon)(B^\varepsilon_h T^h_\gamma) = C_{\beta ah} T^h_\gamma - C_{\beta ab} T^b_\gamma = 0,$$

which implies (a). Similarly we obtain

$$C_{ab\varepsilon} T^\varepsilon_\gamma = (C_{abj} B^j_\varepsilon)(B^\varepsilon_h T^h_\gamma) = C_{abh} T^h_\gamma - C_{abc} T^c_\gamma = 0,$$

which implies (b). Q. E. D.

A $GT\Gamma$ (or $GT\Gamma_0$) is a $TM\Gamma$ (or $TM\Gamma_0$) whose *hv*-torsion tensor vanishes, i. e. $Q^i_{jk} = 0$. Then we can state

Corollary 2. 1. 3. *Let the connection* $M\Gamma$ *in consideration be a* $GT\Gamma$ *(resp.* $GT\Gamma_0$*). Then a subspace* M_m *is totally n-parallel with respect to* $IM\Gamma$ *if and only if* M_m *is both pfd-free and totally geodesic and further the following equations* (2. 23) *and* (2. 24) *(resp.* (2. 25) *and* (2. 26)) *hold :*

$$T_{\beta a\gamma} + T_{a\beta\gamma} + 2(C_{\beta ab} T^b_\gamma + P_{\beta a\gamma}) = 0,\ N^b_j N^j_{a|\gamma} = N^a_j N^j_{b|\gamma}, \tag{2.23}$$

$$T_{ab\gamma} + T_{ba\gamma} + 2(C_{ab\varepsilon} T^\varepsilon_\gamma + C_{abc} T^c_\gamma + P_{ab\gamma}) = 0,\ \lambda^a_{b\gamma} = C_b{}^a{}_\gamma, \tag{2.24}$$

$$T_{\beta a\gamma} + T_{a\beta\gamma} + 2P_{\beta a\gamma} = 0,\ N^b_j N^j_{a|\gamma} = N^a_j N^j_{b|\gamma}, \tag{2.25}$$

$$T_{ab\gamma} + T_{ba\gamma} + 2(C_{abc} T^c_\gamma + P_{ab\gamma}) = 0,\ \lambda^a_{b\gamma} = C_b{}^a{}_\gamma = 0, \tag{2.26}$$

where $T_{\beta a\gamma} = T_{jik} N^i_a B^{jk}_{b\gamma}$, $P_{\beta a\gamma} = P_{jik} N^i_a B^{jk}_{\beta\gamma}$, $T_{ab\gamma} = T_{ijk} N^i_a N^j_b B^k_\gamma$, $P_{ab\gamma} = P_{ijk} N^i_a N^j_b B^k_\gamma$, $T_{jik} = g_{is} T^s_{jk}$ *and* $P_{jik} = g_{is} P^s_{jk}$.

Proof. For either of connections $GT\Gamma$ and $GT\Gamma_0$, we have $y^i g_{ij|k} = 0$. Therefore, contracting (2. 4) by y^β, we get $H^a_\gamma = 0$ (M_m: totally geodesic) and hence $H_\beta{}^a{}_\gamma = 0$ because of Lemma 2. 2. In this case, from (2. 4) and (2. 9) we obtain

$$\begin{aligned} g_{jk|\gamma} B^j_\beta N^k_a &= -(T_{\beta a\gamma} + T_{a\beta\gamma} + 2C_{\beta ah} T^h_\gamma + 2P_{\beta a\gamma}) = 0, \\ g_{jk|\gamma} N^j_a N^k_b &= -(T_{ab\gamma} + T_{ba\gamma} + 2C_{abh} T^h_\gamma + 2P_{ab\gamma}) = 0. \end{aligned} \tag{2.27}$$

On the other hand, from (2. 5) we get $C_\beta{}^a{}_\gamma = 0$ (resp. $2C_{\beta a\gamma} = 0$) (M_m : pfd-free). Applying Lemma 2. 3 to (2. 27) and taking account of Theorem 2. 1, we can deduce

Corollary 2. 1. 3. Q. E. D.

Note 2. 5. Practical examples for Corollary 2. 1. 3 are $H\Gamma$, $IS\Gamma$ and $B\Gamma$, $IS\Gamma_0$.

§ 3. Totally ncd-free subspaces. In this section, we shall be concerned with subspaces that correspond to totally umbilical subspaces in Riemannian geometry.

Let $f(u^\alpha, y^\alpha)$ be a scalar on M_m. Then it is said that the scalar f is *direct-free* if it is independent of y^α.

We shall call a point (u^α) of M_m an *ncd-free* (resp. *nc-constant*) *point* if the following relation holds at the point (u^α) for direct-free scalars f^a (resp. constants f^a):

$$(3.1)\qquad y^\beta H^a_\beta = H^a_o = \bar{L}^2 f^a \ (a = m+1, \cdots\cdots, n).$$

In this case, the square of the normal curvature $N(u^\alpha, y^\alpha)$ in y^α-direction at the point (u^α) is given by $N^2 = \delta_{ab} f^a f^b$. Therefore the normal curvature at an ncd-free (resp. nc-constant) point is direct-free (resp. constant).

We shall say that M_m *is totally ncd-free* (resp. *nc-constant*) if every point of M_m is an ncd-free (resp. nc-constant) point.

In the following, we assume that M_n is endowed with a geo-path connection $M\Gamma$. In this case, it is known [6] that the induced connection $IM\Gamma$ is also a geo-path connection on M_m, and that the following relation holds:

$$(3.2)\qquad T^a{}_o = T^a{}_\gamma y^\gamma = 0,\ H^a_o = \overset{b}{H}{}^a_o := N^a_i(B^i_{oo} + 2G^i).$$

If we differentiate (3. 1) by y^β on making use of (3. 2) and divide the result by 2, then we have

$$(3.3)\qquad \overset{b}{H}{}^a_\gamma + \frac{1}{2}\lambda^{\ a}_{b\gamma}\overset{b}{H}{}^b_o = f^a y_\gamma,$$

where $\overset{b}{H}{}^a_\gamma = N^a_i(B^i_{o\gamma} + G^i{}_k B^k{}_\gamma)$ and $y_\gamma = \bar{L}\partial\bar{L} / \partial y^\gamma (= g_{\beta\gamma} y^\beta)$.

Further if we differentiate (3. 3) by y^β, then we obtain

$$(3.4)\qquad \overset{b}{H}{}_\beta{}^a{}_\gamma = f^a g_{\beta\gamma} - (\lambda^{\ a}_{b\beta}\overset{b}{H}{}^b_\gamma + \lambda^{\ a}_{b\gamma}\overset{b}{H}{}^b_\beta) - \frac{1}{2}\overset{b}{H}{}^b_o(\lambda^{\ a}_{b\gamma\|\beta} + \lambda^{\ a}_{c\gamma}\lambda^{\ c}_{b\beta}),$$

where $\overset{b}{H}{}_\beta{}^a{}_\gamma = N^a_i(B_\beta{}^i{}_\gamma + G^i_{jk}B^{j\,k}_{\beta\gamma})$. By virtue of (3. 1) and (3. 3), the expressions (3. 3) and (3. 4) are expressible in

$$(3.3)'\qquad \overset{b}{H}{}^a_\gamma = f^a y_\gamma - \frac{1}{2}\bar{L}^2\lambda^{\ a}_{b\gamma}f^b,$$

$$\overset{b}{H}_{\beta}{}^{a}{}_{\gamma} = f^a g_{\beta\gamma} - f^b(\lambda^{\ a}_{b\beta} y_\gamma + \lambda^{\ a}_{b\gamma} y_\beta) - \frac{1}{2}\bar{L}^2 f^b(\lambda^{\ a}_{b\gamma \| \beta} - \lambda^{\ a}_{c\beta}\lambda^{\ c}_{b\gamma}) \tag{3.4)'$$

Conversely if we contract (3.3) (or (3.3)') and (3.4) (or (3.4)') by y^β, then we get (3.1) and (3.3) (or (3.3)'). Hence we can state

Theorem 3.1. *Let M_n be endowed a geo-path connection $M\Gamma$. Then the following facts mutually equivalent* :

(a) *M_m is totally ncd-free (resp. nc-costant).*

(b) *For direct-free scalars f^a (resp. constants f^a), the equation* (3.3) (*or* (3.3)') *holds on M_m.*

(c) *For direct-free scalars f^a (resp. constants f^a), the equation* (3.4) (*or* (3.4)') *holds on M_m.*

In the following, we shall consider only a $T\Gamma$ (or $T\Gamma_o$) and the induced connection $IT\Gamma$ (or $IT\Gamma_o$). In this case, since $\widetilde{C}^{i}_{jk} = C^{i}_{jk}$ or $\widetilde{C}^{i}_{jk} = 0$, we first have $\widetilde{S}_{okh} = 0$. Further we have

$$H_{\beta}{}^{a}{}_{\gamma} = \overset{b}{H}_{\beta}{}^{a}{}_{\gamma} + T_{\beta}{}^{a}{}_{\gamma} + Q_{\beta}{}^{a}{}_{\gamma} + [C_{\beta}{}^{a}{}_{b} H^{b}_{\gamma}]^{4)}, \quad T_{\beta}{}^{a}{}_{\gamma} := T^{i}_{jk} N^{a}_{i} B^{jk}_{\beta\gamma}. \tag{3.5}$$

Contracting (3.5) by y^β or by y^γ, we obtain

$$H_{o}{}^{a}{}_{\gamma} = \overset{b}{H}{}^{a}_{\gamma} + T^{a}{}_{\gamma} + D^{a}{}_{\gamma}, \tag{3.6}$$

$$H_{\beta}{}^{a}{}_{o} = \overset{b}{H}{}^{a}_{\beta} - T^{a}{}_{\gamma} + Q_{\beta}{}^{a}{}_{o} + [C_{\beta}{}^{a}{}_{b} \overset{b}{H}{}^{b}_{o}]. \tag{3.7}$$

For the sake of simplicity, we impose the following assumption (called the *TDQ-condition*) :

$$T^{a}{}_{\gamma} = 0, \quad D^{a}{}_{\gamma} = 0, \quad Q_{\beta}{}^{a}{}_{o} = 0. \tag{3.8}$$

In this case, we can state

Lemma 3.1. *The induced connections on M_m from the following connections satisfy the TDQ-condition* :

(a) *$TM\Gamma$: $C\Gamma$, $H\Gamma$, $AMB\Gamma$, $AMC\Gamma$, $AMR\Gamma$; $TM\Gamma_o$: $R\Gamma$, $B\Gamma$, $AMB\Gamma_o$, $AMC\Gamma_o$, $AMR\Gamma_o$,*

(b) *$TMD\Gamma$: $AMD\Gamma$, $AMBD\Gamma$, $AMCD\Gamma$, $MD\Gamma$; $TMD\Gamma_o$: $AMD\Gamma_o$, $AMBD\Gamma_o$, $AMCD\Gamma_o$, $MD\Gamma_o$*

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4) In the following, the terms within the square brackts [ ] vanish for the induced connection $IT\Gamma_o$.

Differentiating the first in (3. 8) by y^{β}, we have

$$T^{a}{}_{\gamma\|\beta} = \lambda^{a}_{b\beta} T^{b}{}_{\gamma} + T_{\beta}{}^{a}{}_{\gamma} = 0,$$

from which it follows that

(3. 9) $$T_{\beta}{}^{a}{}_{\gamma} = 0.$$

Applying (3. 8) and (3. 9) to (3. 5) $\sim$ (3. 7), we have

(3. 10) $$H_{\beta}{}^{a}{}_{\gamma} = \overset{b}{H}_{\beta}{}^{a}{}_{\gamma} + Q_{\beta}{}^{a}{}_{\gamma} + [C_{\beta}{}^{a}{}_{b} \overset{b}{H}{}^{b}_{\gamma}],$$

(3. 11) $$H_{o}{}^{a}{}_{\gamma} = H^{a}_{\gamma} = \overset{b}{H}{}^{a}_{\gamma}, \quad H_{\beta}{}^{a}{}_{o} = \overset{b}{H}{}^{a}_{\beta} + [C_{\beta}{}^{a}{}_{b} \overset{b}{H}{}^{b}_{o}].$$

If we apply (3. 10) and (3. 11) to (1. 18), then we obtain

(3. 12) $$\begin{aligned} \widetilde{R}_{o\delta o\gamma} = {} & \widetilde{R}_{oioh} B^{ih}_{\delta\gamma} + \delta_{ab} \{ \overset{b}{H}{}^{a}_{o}(\overset{b}{H}_{\delta}{}^{b}{}_{\gamma} + Q_{\delta}{}^{b}{}_{\gamma}) - \overset{b}{H}{}^{a}_{\gamma} \overset{b}{H}{}^{b}_{\delta} + [\overset{b}{H}{}^{a}_{o} C_{\delta}{}^{b}{}_{c} \overset{b}{H}{}^{c}_{\gamma} \\ & - \overset{b}{H}{}^{a}_{\gamma} C_{\delta}{}^{b}{}_{c} \overset{b}{H}{}^{c}_{o}] \} + (g_{jk|\gamma} B^{j}{}_{\delta} \overset{b}{H}{}^{a}_{o} - g_{jk|\beta} y^{\beta} B^{j}{}_{\delta} \overset{b}{H}{}^{a}_{\gamma}) N^{k}_{a} \\ & + (\widetilde{P}_{oioh} \overset{b}{H}{}^{a}_{\gamma} - \widetilde{P}_{oikh} B^{k}{}_{\gamma} \overset{b}{H}{}^{a}_{o}) B^{i}{}_{\delta} N^{h}_{a}. \end{aligned}$$

Let M_{m} be totally ncd-free (or nc-constant). Then, on making use of (3. 1), (3. 3)' and a relation $C_{\delta}{}^{b}{}_{c} = C_{\delta}{}^{c}{}_{b}$, we first obtain

(3. 13) $$\begin{aligned} \delta_{ab}(\overset{b}{H}{}^{a}_{o} C_{\delta}{}^{b}{}_{c} \overset{b}{H}{}^{c}_{\gamma} - \overset{b}{H}{}^{a}_{\gamma} C_{\delta}{}^{b}{}_{c} \overset{b}{H}{}^{c}_{o}) &= \frac{1}{2} \bar{L}^{4} \delta_{ab} C_{\delta}{}^{b}{}_{c} (\lambda^{a}_{d\gamma} f^{c} - \lambda^{c}_{d\gamma} f^{a}) f^{d} \\ &= \frac{1}{2} \bar{L}^{4} (\sum_{c} C_{\delta}{}^{c}{}_{b} \lambda^{b}_{d\gamma} f^{c} f^{d} - \delta_{ab} C_{\delta}{}^{b}{}_{c} \lambda^{c}_{d\gamma} f^{a} f^{d}) = 0. \end{aligned}$$

On the other hand, from (1. 12) we have

(3. 14) $$\delta_{ab} \lambda^{a}_{c\gamma} f^{b} f^{c} = C_{ab\gamma} f^{a} f^{b}, \quad \delta_{ab} \lambda^{b}_{c\gamma\|\delta} f^{a} f^{c} = C_{ab\gamma\|\delta} f^{a} f^{b}.$$

If we apply (3. 1), (3. 3)', (3. 4)' and (3. 13) to (3. 12) and use (3. 14), then we obtain

(3. 15) $$\frac{1}{2}(\widetilde{R}_{o\delta o\gamma} + \widetilde{R}_{o\gamma o\delta}) = \frac{1}{2}(\widetilde{R}_{oioh} + \widetilde{R}_{ohoi}) B^{ih}_{\delta\gamma} + \bar{L}^{2} N^{2} h_{\delta\gamma} + \frac{1}{2} \Phi_{\delta\gamma},$$

where

$$\begin{aligned} \Phi_{\delta\gamma} = {} & \bar{L}^{2} \{ f_{a}(Q_{\delta}{}^{a}{}_{\gamma} + Q_{\gamma}{}^{a}{}_{\delta}) - (C_{ab\delta} y_{\gamma} + C_{ab\gamma} y_{\delta}) f^{a} f^{b} - \bar{L}^{2}(C_{ab\gamma\|\delta} f^{a} f^{b} - \\ & f_{a} \lambda^{a}_{b\delta} \lambda^{b}_{c\gamma} f^{c} + \frac{1}{2} \sum_{a} \lambda^{a}_{b\delta} \lambda^{a}_{c\gamma} f^{b} f^{c}) \} + \bar{L}^{2} N^{k}_{a} f^{a} \{ (g_{jk|\delta} B^{j}{}_{\gamma} + g_{jk|\gamma} B^{j}{}_{\delta}) \end{aligned}$$

(3.16) $$- \widetilde{P}_{ojhk}(B^{j\,h}_{\delta\,\gamma} + B^{j\,h}_{\gamma\,\delta})\} - (g_{jk|\beta}y^{\beta} - \widetilde{P}_{ojok})N^k_a\{f^a(B^j{}_\delta y_\gamma + B^j{}_\gamma y_\delta)$$

$$- \frac{1}{2}\bar{L}^2(B^j{}_\delta \lambda^a_{b\gamma} + B^j{}_\gamma \lambda^a_{b\delta})f^b\},\ \text{being } f_a = \delta_{ab}f^b.$$

Consequently we can state

Theorem 3.2. *Suppose that M_n is endowed with a $T\Gamma$ (or $T\Gamma_o$) and the induced connection $IT\Gamma$ (or $IT\Gamma_o$) satisfies the TDQ-condition, and that the tensor $\Phi_{\delta\gamma}$ defined by* (3.16) *vanishes. Then if M_n is of scalar curvature R (resp. of costant curvature R) with respect to $T\Gamma$ (or $T\Gamma_o$) and M_m is totally ncd-free (resp. nc-constant) with N, then M_m is of scalar curvature $(R+N^2)$ (resp. of constant curvature $(R+N^2)$) with respect to $IT\Gamma$ (or $IT\Gamma_o$).*

By virtue of (3.1) and (3.3)', we have

$$g_{jk|\delta} = g_{jk|h}B^h{}_\delta + [2C_{jkc}f^c y_\delta - \bar{L}^2 C_{jka}\lambda^a_{b\delta}f^b],$$

$$g_{jk|\beta}y^\beta = g_{jk|h}y^h + [2\bar{L}^2 C_{jkc}f^c],$$

from which it follows that

(3.17) $$\begin{aligned}&\bar{L}^2 N^k_a f^a g_{jk|\delta}B^j{}_\gamma - g_{jk|\beta}y^\beta N^k_a(f^a B^j{}_\gamma y_\delta - \frac{1}{2}\bar{L}^2 B^j{}_\gamma \lambda^a_{b\delta}f^b)\\ &= \bar{L}^2 N^k_a f^a g_{jk|h}B^{j\,h}_{\delta\,\gamma} - g_{jk|h}y^h N^k_a(f^a B^j{}_\gamma y_\delta - \frac{1}{2}\bar{L}^2 B^j{}_\gamma \lambda^a_{b\delta}f^b).\end{aligned}$$

Applying (3.17) to (3.16). we obtain

(3.18) $$\Phi_{\gamma\delta} = U_{\gamma\delta} + V_{\gamma\delta},$$

where

(3.19) $$\begin{aligned}U_{\gamma\delta} = \bar{L}^2\{&\bar{L}^2(f_a\lambda^a_{b\gamma}\lambda^b_{c\delta}f^c - C_{ab\delta\|\gamma}f^a f^b - \frac{1}{2}\sum_a \lambda^a_{b\gamma}\lambda^a_{c\delta}f^b f^c)\\ &- (C_{ab\gamma}y_\delta + C_{ab\delta}y_\gamma)f^a f^b\},\end{aligned}$$

(3.20) $$\begin{aligned}V_{\gamma\delta} = \bar{L}^2\{&f_a(Q_\gamma{}^a{}_\delta + Q_\delta{}^a{}_\gamma) + (g_{jk|h} - \widetilde{P}_{ojhk})N^k_a f^a(B^{j\,h}_{\gamma\,\delta} + B^{j\,h}_{\delta\,\gamma})\}\\ &- (g_{jk|o} - \widetilde{P}_{ojok})N^k_a\{f^a(B^j{}_\gamma y_\delta + B^j{}_\delta y_\gamma) - \frac{1}{2}\bar{L}^2(B^j{}_\gamma\lambda^a_{b\delta} + B^j{}_\delta\lambda^a_{b\gamma})f^b\},\end{aligned}$$

being $g_{jk|o} = g_{jk|h}y^h$.

Suppose that the following condition holds:

(3.21) $$f_a\lambda^a_{b\gamma} = 0,\quad \lambda^a_{b\gamma}f^b = 0.$$

Then from (1. 12) we have $C_{ab\gamma}f^a = 0$. Therefore it is seen that the tensor $U_{\gamma\delta}$ defined by (3. 19) vanishes. In this case, if the connection $T\Gamma$ (or $T\Gamma_o$) is h-metrical, then from (3. 20) we have

$$V_{\gamma\delta} = \{\bar{L}^2(Q_{\gamma a\delta} + Q_{\delta a\gamma} - \widetilde{P}_{o\gamma\delta a} - \widetilde{P}_{o\delta\gamma a}) + \widetilde{P}_{o\gamma o a}y_\delta + \widetilde{P}_{o\delta o a}y_\gamma\}f^a, \tag{3. 22}$$

where $Q_{\gamma a\delta} = Q_{jik}N^i_a B^{jk}_{\gamma\delta}$, $\widetilde{P}_{o\gamma\delta a} = \widetilde{P}_{ojhk}N^k_a B^{jh}_{\gamma\delta}$ and $\widetilde{P}_{o\gamma o a} = \widetilde{P}_{ojok}N^k_a B^j{}_\gamma$.

Consequently we can state

Corollary 3. 2. 1. *Suppose that M_n is endowed with an h-metrical $T\Gamma$ (or $T\Gamma_o$) and the induced connection $IT\Gamma$ (or $IT\Gamma_o$) satisfies the TQD-condition, and that the condition* (3. 21) *holds and the tensor $V_{\gamma\delta}$ defined by* (3. 22) *vanishes. Then if M_n is of scalar curvature R (resp. of constant curvature R) with respect to $T\Gamma$ (or $T\Gamma_o$) and M_m is totally ncd-free (resp. nc-constant) with N, then M_m is of scalar curvature $(R+N^2)$ (resp. of constant curvature $(R+N^2)$) with respect to $IT\Gamma$ (or $IT\Gamma_o$).*

Note 3. 1. In the above Corollary, if the $T\Gamma$ (or $T\Gamma_o$) is an h-metrical $TM\Gamma$ (or $TM\Gamma_o$) then the tensor $V_{\gamma\delta}$ in (3. 22) is expressed in

$$V_{\gamma\delta} = \{\bar{L}^2(Q_{\gamma a\delta} + Q_{\delta a\gamma} + Q_{a\gamma\delta} + Q_{a\delta\gamma}) - (Q_{a\delta o}y_\gamma + Q_{a\gamma o}y_\delta)\}f^a. \tag{3. 23}$$

where $Q_{a\delta o} = Q_{jik}N^j_a B^i{}_\delta y^k$. In particular, we have

$$\begin{aligned} V_{\gamma\delta} &= -4\bar{L}^2 f^a P_{\gamma a\delta} \quad \text{for } C\Gamma \text{ (or } R\Gamma\text{)}, \\ V_{\gamma\delta} &= -4\bar{L}^2 f^a(f\bar{L}C_{\gamma a\delta} + P_{\gamma a\delta}) \quad \text{for } AMR\Gamma \text{ (or } AMR\Gamma_o\text{)}, \end{aligned} \tag{3. 24}$$

where f is a function of $x^i(u^a)$ alone.

If the connection $T\Gamma$ (or $T\Gamma_o$) is a $GT\Gamma$ (or $GT\Gamma_o$), then we have

$$\begin{aligned} g_{jk|h} &= -(T_{jkh} + T_{kjh} + 2C_{jks}T^s{}_h + 2P_{jkh}), \\ g_{jk|o} &= T_{jk} + T_{kj}, \ \widetilde{P}_{ojkh} = -Q_{hjk} = 0. \end{aligned} \tag{3. 25}$$

Applying (3. 25) to (3. 20) and using (3. 8), (3. 9) and (3. 21), we get

$$\begin{aligned} V_{\gamma\delta} = &-\bar{L}^2 f^a\{T_{a\gamma\delta} + T_{a\delta\gamma} + 4P_{\gamma a\delta} + 2(C_{\gamma a\beta}T^\beta{}_\delta + C_{\delta a\beta}T^\beta{}_\gamma)\} \\ &- f^a(T_{\gamma a}y_\delta + T_{\delta a}y_\gamma). \end{aligned} \tag{3. 26}$$

Hence we can state

Corollary 3. 2. 2. *Suppose that M_n is endowed with a $GT\Gamma$ (or $GT\Gamma_o$) and the* *induced connection $IGT\Gamma$ (or $IGT\Gamma_o$) satisfies the TDQ-condition, and that the condition* (3.21) *holds and the tensor $V_{\gamma\delta}$ defined by* (3.26) *vanishes. Then if M_n is of scalar curvature R (resp. of constant curvature R) with respect to $GT\Gamma$ (or $GT\Gamma_o$) and M_m is totally ncd-free (resp. nc-constant) with N, then M_m is of scalar curvature ($R + N^2$) (resp. of constant curvature ($R + N^2$)) with respect to $IGT\Gamma$ (or $IGT\Gamma_o$).*

We shall call a GT(resp. $GT(0)$)-connection a GTA (resp. $GTA(0)$) *-connection* and denote it by $GTA\Gamma$(resp. $GTA\Gamma_o$) if the tensor $T^i{}_k$ is defined by $T^i{}_k = fLh^i{}_k$.

Note 3.2. As practical examples for Corollary 3.2.2, we have

$$V_{\gamma\delta} = -4\bar{L}^2 f^a P_{\gamma a\delta} \qquad \text{for } H\Gamma (\text{or } B\Gamma),$$

(3.27)

$$V_{\gamma\delta} = -4\bar{L}^2 f^a (f\bar{L}C_{\gamma a\delta} + P_{\gamma a\delta}) \qquad \text{for } GTA\Gamma (\text{or } GTA\Gamma_o),$$

where f is a function of $x^i(u^a)$ alone.

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