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A THEORY OF SUBSPACES IN A FINSLER SPACE

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Introduction. In Riemannian geometry, as is well known, the theory of subspaces has been develoved through the Riemannian connection and the induced Riemannian one. Because the Riemannian connection is the best for investigating this geometry and the induced Riemannian connection coincides with the intrinsic one.

In Finsler geometry, the circumstances stated above are entirely different. On Finsler spaces, various connections are defined and the properties of them are discussed (for example, [1],[1] [3], [5], [9], [10], [11], [13], [15], [17], [18], [19], [23], [26], [30], [31] and [32] etc.). It seems, to this auther, that no one can, for the present, say which connection is best for the investigation of Finsler geometry.

Next, the theories of subspaces (including hypersurfaces) have been studied by many authors from their own stand-points (for example, [2], [4], [6], [7], [8], [12], [14], [16], [20], [21], [22], [24], [25], [26], [27], [29], [30], [33], [34] and [35] etc.). Except special cases ([6], [20] and [21]), any induced connections treated by every author do not, in general, coincide with the intrinsic connections. May-be this fact will be inevitable.

As is well known, a Finsler space M is a metrical space endowed with Finsler metric. In the present paper, we consider three kinds of connections on M. The first is a Matsumoto connection, which is a quite general connection with no metrical property. The second is a TMD(or $TMD(0)$)-connection, which is a less general connection with some geometrical properties. The third is a TM(or TM

1) Numbers in brackets refer to the references at the end of the paper.

(0))-connection, which is a slight generalization of the typical connections (for example, the Cartan connection or the Berwald).

The principal purpose of the present paper is to construct a theory of subspaces of M through the above connections. In § 1, a Matsumoto connection is defined and the various axioms to be imposed on it are listed. The remaining two kinds of connections are characterized by several axioms and many their special connections are considered. In § 2, a subspace of M is considered and various quantities and relations are discussed. In § 3 and § 4, we introduce the induced Matsumoto connection and study the various properties of it. And further we derive the generalized Gauss and Codazzi equations. Two sections § 5 and § 6 are devoted to the investigations of the induced TM(or $TM(0)$)-connection and of the induced TMD(or $TMD(0)$)-connection. In the last section § 7, the various special subspace are investigated. For example, totally geodesic subspaces and totally ncd-free subspaces etc.

The terminologies and notations refer to papers [30] ~ [35] unless otherwise stated.

§ 1. Preliminaries. Let M_n be an n-dimensional Finsler space with a fundamental function $L(x, y)$, and endowed with a Matsumoto connection $M\Gamma=(\Gamma^i_{jk}, \Gamma^i_k, \tilde{C}^i_{jk})$ ([31], [32]). Then we have three tensors T^i_k, Q^i_{jk} and $\tilde{C}^i_{jk}$ on M_n such that

(1.1)
(a) T^i_k is a $(1)p$-homogeneous[2] (1, 1)-tensor.
(b) Q^i_{jk} is a $(0)p$-homogeneous (1, 2)-tensor.
(c) $\tilde{C}^i_{jk}$ is a $(-1)p$-homogeneous (1, 2)-tensor.

And further the non-linear connection and the h-connection are given as follows :

$$\Gamma^i_k = G^i_k + T^i_k, \tag{1.2}$$

where G^i_k is the non-linear connection of Cartan (or Berwald).

$$\Gamma^i_{jk} = \Gamma^i_{k1j} + Q^i_{jk} = G^i_{jk} + T^i_{jk} + Q^i_{jk}, \tag{1.3}$$

where $\Gamma^i_{k1j} = \partial\Gamma^i_k / \partial y^j$, $T^i_{jk} = T^i_{k1j}$ and $G^i_{jk} (= G^i_{k1j})$ is the h-connection of Berwald.

Note 1.1. Given three tensors satisfying (1, 1), a Matsumoto connection

2) "(r)p-homogeneous" means "positively homogeneous of degree r in y^i".

$M\Gamma$ is uniquely determined. The $M\Gamma$ is one of the most general connections on M_n.

The fundamental tensor and the C-tensor are as follows:

$$(1.4)\quad \begin{aligned} &\text{(a)}\ \ g_{ij} = \tfrac{1}{2} L^2{}_{(i)(j)} \ , \\ &\text{(b)}\ \ C_{ijk} = \tfrac{1}{2} g_{ij(k)} \ , \quad C_j{}^i{}_k = g^{ih} C_{jhk} \ , \end{aligned}$$

where g^{ih} is the reciprocal tensor of g_{ij}.

A Matsumoto connection $M\Gamma$ has no other geometrical property than a connection on M_n. Therefore we need to consider the various axioms to be imposed on $M\Gamma$. We shall list them extensively.

(F1) $M\Gamma$ is metrical, i.e., $L_{|k} = 0$.

(F2) The deflexion tensor $D^i{}_k$ vanishes, i.e., $D^i{}_k = y^j \Gamma_j{}^i{}_k - \Gamma^i{}_k = 0$.

(F3) $M\Gamma$ is v-metrical and v-symmetric, i.e., $\overline{C}_j{}^i{}_k = C_j{}^i{}_k$.

(F3)$_1$ $M\Gamma$ is v-natural, i.e., $\overline{C}_j{}^i{}_k = 0$.

(F3)$_2$ $M\Gamma$ is v-semi-symmetric, that is, the v-torsion tensor is

$$(1.5)\qquad \overline{S}_j{}^i{}_k (= \overline{C}_j{}^i{}_k - \overline{C}_k{}^i{}_j) = \delta^i{}_j t_k - \delta^i{}_k t_j ,$$

where t_k is a $(-1)p$-homogeneous vector.

(F4) With respect to $M\Gamma$, the absolute differential Dy_i of $y_i (= g_{ij} y^j)$ is given by $Dy_i = g_{ij} Dy^j$ (or equivalently $y^i Dg_{ij} = 0$).

(F5) Paths with respect to $M\Gamma$ are always geodesics of M_n.

(F6) $M\Gamma$ is h-metrical, i.e., $g_{ij|k} = 0$.

(F7) $M\Gamma$ is h-symmetric, that is, the h-torsion tensor $\tau_j{}^i{}_k (= \Gamma_j{}^i{}_k - \Gamma_k{}^i{}_j)$ vanishes.

(F8) The hv-torsion tensor $\overline{P}^i{}_{kj} (= -Q_j{}^i{}_k)$ of $M\Gamma$ vanishes.

(F9) $M\Gamma$ is h-semi-symmetric, that is, the h-torsion tensor is

$$(1.6)\qquad \tau_j{}^i{}_k = \delta^i{}_j s_k - \delta^i{}_k s_j ,$$

where s_k is a $(0)p$-homogeneous vector.

(F10) The non-linear connection of $M\Gamma$ is given by $G^i{}_k$ (or equivalently $T^i{}_k = 0$).

Note 1.2. Most of the connections used hitherto may be characterized by choosing suitable axioms in (F1) ~ (F10).

A Matsumoto connection $M\Gamma$ is called a TM(resp. $TM(0)$)-connection if it is characterized by five axioms (F1), (F2), (F3) (resp. $(F3)_1$), (F4) and (F5). It is known ([30], [31]) that a TM(resp. $TM(0)$)-connection is uniquely determined if we have three tensors satisfying (1. 1) and

$$(1.7) \qquad T^o{}_k = T^i{}_o = 0, \quad Q_j{}^o{}_k = Q_o{}^i{}_k = 0, \quad \tilde{C}_j{}^i{}_k = C_j{}^i{}_k \; (resp.\ \tilde{C}_j{}^i{}_k = 0),$$

where the index 0 indicates contraction of a tensor, for emxaple $T^i{}_k$, by y_i (or y^k).

Typical connections on M_n, that is, the Berwald connection $B\Gamma$, the Hashiguchi $H\Gamma$, the Rund $R\Gamma$ and the Cartan $C\Gamma$ are all special TM(or $TM(0)$)-connections, and the tensors determing them are as follows :

(1. 8)

(a) $T^i{}_k = 0, \; Q_j{}^i{}_k = 0, \; \tilde{C}_j{}^i{}_k = 0$ for $B\Gamma$,

(b) $T^i{}_k = 0, \; Q_j{}^i{}_k = 0, \; \tilde{C}_j{}^i{}_k = C_j{}^i{}_k$ for $H\Gamma$,

(c) $T^i{}_k = 0, \; Q_j{}^i{}_k = -P_j{}^i{}_k, \; \tilde{C}_j{}^i{}_k = 0$ for $R\Gamma$,

(d) $T^i{}_k = 0, \; Q_j{}^i{}_k = -P_j{}^i{}_k, \; \tilde{C}_j{}^i{}_k = C_j{}^i{}_k$ for $C\Gamma$,

where $P_j{}^i{}_k$ is the hv-torsion tensor with respect to $C\Gamma$.

An h-metrical TM-connection is called an RTM-connection. It is known [30] that this connection is characterized by four axioms (F2), (F3), (F5) and (F6), and that it is uniquely determined if three tensors $T^i{}_k$, $\tilde{C}_j{}^i{}_k$ and Z_{ijk} are given as follows :

(1. 9)

(a) $T^i{}_k$ satisfies (1. 7). (b) $\tilde{C}_j{}^i{}_k = C_j{}^i{}_k$.

(c) Z_{ijk} is a $(0)p$-homogeneous $(0, 3)$-tensor such that

$$Z_{ijk} + Z_{jik} = 0, \quad Z_{ojk} = Z_{iok} = 0.$$

In this case, the h-connection is expressible in

$$(1.10) \qquad \Gamma_j{}^i{}_k = \Gamma^*{}_j{}^i{}_k - C_j{}^i{}_r T^r{}_k + \tfrac{1}{2} g^{ir} (Z_{jrk} + T_{jrk} - T_{rjk}),$$

where $\Gamma^*{}_j{}^i{}_k$ is the h-connection of $C\Gamma$ and $T_{jrk} = g_{rh} T_j{}^h{}_k$.

The Cartan connection $C\Gamma$, the IS-connection $IS\Gamma$ and an AMR-connection $AMR\Gamma$ are all RTM-connections, and two tensors $T^i{}_k$ and Z_{ijk} are given as follows :

(1. 11)
(a) For $C\Gamma$, $T^i{}_k = 0$ and $Z_{ijk} = 0$.

(b) For $IS\Gamma$, $Z_{ijk} = 0$ and $T^i{}_k$ is defined by

$$T_{ijk} + T_{jik} + 2(C_{ijr}\, T^r{}_k + P_{ijk}) = 0.$$

(c) For $AMR\Gamma$, $T^i{}_k = f(x, y)\, Lh^i{}_k$ and $Z_{ijk} = L\,(f_{\|j}\, h_{ik} - f_{\|i}\, h_{jk})$,

where $f(x, y)$ is a $(0)p$-homogeneous scalar and $h^i{}_k = \delta^i{}_k - l^i\, l_k$.

In this case, the h-connections of $IS\Gamma$ and $AM\Gamma$ are expressible in

(1. 12) $$\Gamma^i_{j\,k} = G^{\,i}_{j\,k} + T^i_{j\,k},$$

(1. 13) $$\Gamma^i_{j\,k} = \Gamma^{*i}_{\,j\,k} + f(l_j\, h^i{}_k - l^i\, h_{jk} - LC^{\,i}_{j\,k}).$$

Note 1.3. The hv-curvature tensor $\widetilde{P}_j{}^i{}_{kh}$ with respect to $IS\Gamma$ always vanishes and hence $Q_h{}^i{}_k (= -y^j\, \widetilde{P}^i_{j\,kh}) = 0$.

Note 1.4. A Wagner connection is characterized by four axioms (F2), (F3), (F6) and (F9) [12]. Therefore an $AMR\Gamma$ is a special Wagner connection satisfying (F5). In particular if the scalar $f(x, y)$ is defined by $f = LC^i{}_{|i} / (n-1)$ ($C^i = C_j{}^i{}_k\, g^{jk}$), then this $AMR\Gamma$ is called the Barthel-Matsumoto connection ([18], [32]).

An h-symmetric TM(resp. $TM(0)$)-connection is called an STM(resp. $STM(0)$)-connection and denoted by $STM\Gamma$(resp. $STM\Gamma_o$). It is known [30] that if we have a tensor $Q_j{}^i{}_k$ satisfying (b) in (1. 1) then an $STM\Gamma$(or $STM\Gamma_o$) is uniquely determined and $\Gamma^i{}_k$, $\Gamma^i_{j\,k}$ are expressible in

(1. 14) $$\Gamma^i{}_k = G^{\,i}{}_k + T^{\,i}{}_k = \tfrac{1}{2} Q^{\,i}_{k\,o},$$

(1. 15) $$\Gamma^i_{j\,k} = G^{\,i}_{j\,k} + \tfrac{1}{2}(Q^{\,i}_{j\,k} + Q^{\,i}_{k\,j}) + \tfrac{1}{4}(Q^{\,i}_{k\,o\|j} + Q^{\,i}_{j\,o\|k}).$$

An AMB(or $AMB(0)$)-connection $AMB\Gamma$(or $AMB\Gamma_o$) and an AMC(or $AMC(0)$)-connection $AMC\Gamma$(or $AMC\Gamma_o$) are special STM(or $STM(0)$)-connections, and tensors determing them are as follows :

(1. 16) $$Q_j{}^i{}_k = 2fl_k\, h^i{}_j - Lf_{\|j}\, h^i{}_k \qquad \text{for } AMB\Gamma \text{(or } AMB\Gamma_o\text{)},$$

(1. 17) $$Q_j{}^i{}_k = 2fl_k\, h^i{}_j - Lf_{\|j}\, h^i{}_k - P^i_{j\,k} \qquad \text{for } AMC\Gamma \text{(or } AMC\Gamma_o\text{)}.$$

In this case, the non-linear connections are commonly given by

(1.18) $\Gamma^i_k = G^i_k + T^i_k,\quad T^i_k = fLh^i_k,$

and the h-connections are respectively expressible in

(1.19) $\Gamma^i_{jk} = G^i_{jk} + f(l_j h^i_k + l_k h^i_j - l^i h_{jk}),$

(1.20) $\Gamma^i_{jk} = \Gamma^{*i}_{jk} + f(l_j h^i_k + l_k h^i_j - l^i h_{jk}).$

A Matsumoto connection $M\Gamma$ is called a *TMD*(resp. *TMD*(0))-connection if it is characterized by four axioms (F1), (F3)(resp. $(F3)_1$), (F4) and (F5), and denoted by $TMD\Gamma$(resp. $TMD\Gamma_o$). It is known ([31], [32]) that a $TMD\Gamma$(resp. $TMD\Gamma_o$) is uniquely determined if three tensor T^i_k, $Q_j{}^i{}_k$ and $\widetilde{C}_j{}^i{}_k$ are given as follows : They satisfy (1. 1) and

(1.21) $T^o_k = T^i_o = 0,\ D_{jk} + Q_{jok} = 0,\ \widetilde{C}_j{}^i{}_k = C_j{}^i{}_k\ (resp.\ \widetilde{C}_j{}^i{}_k = 0),$

where $D^i_k = Q_o{}^i{}_k$ and $Q_{ojk} = D_{jk} = g_{jr} D^r_k$.

A $TMD\Gamma$(resp. $TMD\Gamma_o$) is called an *STD*(rsep. *STD*(0))-connection if it further satisfies (F6) and (F7), and denoted by $STD\Gamma$(resp. $STD\Gamma_o$). It is known [31] that an $STD\Gamma$(resp. $STD\Gamma_o$) is characterized by four axioms (F1), (F3)(resp. $(F3)_1$), (F6) and (F7), and the h-connection is expressible in

(1.22) $\Gamma^i_{jk} = \Gamma^{*i}_{jk} + W^i_{jk},\quad W^i_{jk} = C_{jkr} T^{ri} - C_j{}^i{}_r T^r_k - C_k{}^i{}_r T^r_j,$

where $T^{ri} = g^{is} T^r_s$. In this case, the above connection is uniquely determined if we have a tensor T^i_k satisfying $T^i_o = T^o_k = 0$ with (a) in (1. 1).

An $STD\Gamma$(resp. $STD\Gamma_o$) is called an *AMD*(resp. *AMD*(0))-connection and denoted by $AMD\Gamma$(resp. $AMD\Gamma_o$), if a tensor T^i_k is defined by $T^i_k = fLh^i_k$. The non-linear connection and the h-connection are given by

(1.23) $\Gamma^i_k = G^i_k + fLh^i_k,\ \Gamma^i_{jk} = \Gamma^{*i}_{jk} - fLC^i_{jk}.$

An $STD\Gamma$(resp. $STD\Gamma_o$) is called a *CD*(resp. *RD*)-connection and denoted by $CD\Gamma$(resp. $RD\Gamma$), if a tensor T^i_k is defined by

(1.24) $T^i_k = fL^3 C^i C_k\ (C^i = C^i_{jk} g^{jk},\ C_k = g_{kh} C^h).$

In this case, the non-linear connection and the h-connection are as follows :

(1.25) $\Gamma^i_{\ k} = G^i_{\ k} + fL^3 C^i C_k$,

(1.26) $\Gamma^i_{j\,k} = \Gamma^{*i}_{j\,k} + fL^3 (C_{jkr} C^r C^i - C_j{}^i{}_r C^r C_k - C_k{}^i{}_r C^r C_j)$.

A $TMD\Gamma$(resp. $TMD\Gamma_o$) is called a GQD(resp. $GQD(0)$)-connection and denoted by $GQD\Gamma$(resp. $GQD\Gamma_o$), if it satisfies (F10) instead of (F1).

Given a tensor $Q_j{}^i{}_k$ satisfying (b) in (1. 1) and

(1.27) $D_{jk} (= Q_{ojk}) + Q_{jok} = 0$,

a $GQD\Gamma$(resp. $GQD\Gamma_o$) is uniquely determined, and the non-linear connection and the h-connection are given by

(1.28) $\Gamma^i_{\ k} = G^i_{\ k}$, $\Gamma^i_{j\,k} = \Gamma^{*i}_{j\,k} + W^i_{j\,k}$, $W^i_{j\,k} = Q_j{}^i{}_k + P^i_{j\,k}$.

A $GQD\Gamma$(resp. $GQD\Gamma_o$) is called an MD(resp. $MD(0)$)-connection and denoted by $MD\Gamma$(resp. $MD\Gamma_o$), if it further satisfies (F6) and (F9). If we put $s_j = -f(x, y) \cdot l_j$ in (1. 6), then the h-connection is expressed in

(1.29) $\Gamma^i_{j\,k} = \Gamma^{*i}_{j\,k} + W^i_{j\,k}$, $W^i_{j\,k} = f(l_j \delta^i_{\ k} - l^i g_{jk})$.

Note 1.5. An $MD\Gamma$(or $MD\Gamma_o$) is a special Miron connection.

A $GQD\Gamma$(resp. $GQD\Gamma_o$) is called a $\widetilde{HD}$(resp. $\widetilde{BD}$)-connection and denoted by $\widetilde{HD}\Gamma$(resp. $\widetilde{BD}\Gamma$), if it further satisfies (F7).

A $\widetilde{HD}\Gamma$(resp. $\widetilde{BD}\Gamma$) is a HD(resp. BD)-connection and denoted by $HD\Gamma$(resp. $BD\Gamma$), if the tensor $W^i_{j\,k}$ in (1. 28) is given by

(1.30) $W^i_{j\,k} = fL^2 (l_j C^i C_k + l_k C^i C_j - l^i C_j C_k) + P^i_{j\,k}$.

Note 1.6. An $HD\Gamma$(resp. $BD\Gamma$) is closely similar to $H\Gamma$(resp. $B\Gamma$).

An $\widetilde{HD}\Gamma$(resp. $\widetilde{BD}\Gamma$) is called an $AMBD$(resp. $AMBD(0)$)-connection and an $AMCD$(resp. $AMCD(0)$)-connection respectively if the tensor $W^i_{j\,k}$ is given by (1. 31) and (1. 32) below.

(1.31) $W^i_{j\,k} = f(l_j h^i_{\ k} + l_k h^i_{\ j} - l^i h_{jk})$,

(1.32) $W^i_{j\,k} = f(l_j h^i_{\ k} + l_k h^i_{\ j} - l^i h_{jk}) - P^i_{j\,k}$.

Note 1.6. The above connections are closely similar to the corresponding connections without the letter D. For example, an $AMCD$-connection is closely similar to an AMC-connection.

§ 2. Subspaces of M_n. Let M_m be an m-dimensional subspace of M_n represented parametrically by the equation

$$(2.1) \qquad x^i = x^i(u^\alpha)^{3)} \quad (i=1,2,\ldots,n;\ \alpha=1,2,\ldots,m),$$

where we suppose that the variables u^α form a coordinates system of M_m.

We put

$$(2.2) \qquad B^i_\alpha = \partial x^i / \partial u^\alpha$$

and assume that the matrix (B^i_α) is of rank m.

If we denote the components of a vector X^i tangent to M_m by X^α in terms of the u^α-system, then they are related by

$$(2.3) \qquad X^i = B^i_\alpha X^\alpha.$$

We consider a curve $C : u^\alpha = u^\alpha(t)$ contained in M_m. Then the C is also a curve in M_n represented by $x^i = x^i(u^\alpha(t))$ and the following relation holds because of (2. 2):

$$(2.4) \qquad y^i = dx^i / dt = B^i_\alpha (du^\alpha / dt) = B^i_\alpha y^{\alpha\,4)}$$

The induced metric function $\overline{L}(u^\alpha, y^\alpha)$ on M_m from $L(x^i, y^i)$ on M_n is given by

$$(2.5) \qquad \overline{L}(u^\alpha, y^\alpha) = L(x^i(u^\alpha), B^i_\alpha y^\alpha).$$

The fundamental tensor $g_{\alpha\beta}(u^\alpha, y^\alpha)\ (= \tfrac{1}{2}\partial^2 \overline{L}^2 / \partial y^\alpha \partial y^\beta)$ on M_m is expressible in

$$(2.6) \qquad g_{\alpha\beta}(u^\alpha, y^\alpha) = g_{ij}(x^i, y^i) B^i_\alpha B^j_\beta,$$

The covariant vector y_α corresponding to y^α is expressible in

$$(2.7) \qquad y_\alpha = g_{\alpha\beta} y^\beta = \overline{L}\partial\overline{L} / \partial y^\alpha = y_i B^i_\alpha,$$

differentiation of which by y_β yields

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3) Here and in the following, Latin indices i, j, k, ... run from 1 to n, while Greek indices α, β, γ, ... from 1 to m.

4) If no confusion occurs, then we shall use y^α instead of the usual notation v^α.

(2.8) $$\delta^{\beta}{}_{\alpha} = B^{\beta}{}_{i} B^{i}{}_{\alpha}, \quad B^{\beta}{}_{i} := \partial y_i / \partial y_{\beta} \ [26].$$

Then the reciprocal tensor $g^{\alpha\beta}$ of $g_{\alpha\beta}$ is given by

(2.9) $$g^{\alpha\beta}(u^{\alpha}, y_{\alpha}) = g^{ij}(x^i, y_i) B^{\alpha}{}_{i} B^{\beta}{}_{j}.$$

In this case, the following relations hold :

(2.10) $$B^{\alpha}{}_{i} = g^{\alpha\beta} g_{ij} B^{j}{}_{\beta}, \quad B^{j}{}_{\beta} g^{\alpha\beta} = g^{ij} B^{\alpha}{}_{i},$$
$$B^{i}{}_{\alpha} = g_{\alpha\beta} B^{\beta}{}_{j} g^{ij}, \quad g_{ij} B^{j}{}_{\alpha} = g_{\alpha\beta} B^{\beta}{}_{i}.$$

With respect to a vector y^i in (2.4), we choose $n-m$ normals $N^i{}_a$ $(a = m+1, \ldots, n)$ satisfying the relations

(2.11) $$g_{ij} N^i_a N^j_b = \delta_{ab},^{5)} \quad N^a_i := g_{ij} N^j_a,^{6)} \quad N^a_i B^i{}_{\alpha} = 0.$$

Then we have

(2.12) $$B^{\alpha}{}_{i} N^i_a = 0, \quad B^i{}_{\alpha} B^{\alpha}{}_{j} = \delta^i{}_j - N^i{}_j,$$

(2.13) $$N^i{}_j := \sum_{b=m+1}^{n} N^i_b N^b_j = N^i_b N^b_j.^{7)}$$

From (2.4), (2.7) and (2.11) ~ (2.13) it follows that

(2.14) $$N^a_i y^i = N^i_a y_i = 0, \quad N^i{}_j y_i = N^i{}_j y^j = 0.$$

Hereafter we shall use the following notations :

$$B^{ij\cdots\cdots k}_{\alpha\beta\cdots\cdots\gamma} = B^i_{\alpha} B^j_{\beta} \cdots\cdots B^k_{\gamma}, \quad B^{\alpha\beta\cdots\cdots\gamma}_{ij\cdots\cdots k} = B^{\alpha}_i B^{\beta}_j \cdots\cdots B^{\gamma}_k, \quad B^{i\beta k}_{\alpha j\gamma} = B^i_{\alpha} B^{\beta}_j B^k_{\gamma} \ \text{etc.}$$

Then we have

(2.15) $$g_{ij} = g_{\alpha\beta} B^{\alpha\beta}_{ij} + N_{ij}, \quad g^{ij} = g^{\alpha\beta} B^{ij}_{\alpha\beta} + N^{ij},$$

where $N_{ij} = g_{ik} N^k{}_j$ and $N^{ij} = g^{jk} N^i{}_k$.

The C-tensor $C_{\alpha\beta\gamma}$ on M_m is defined by $C_{\alpha\beta\gamma} = \frac{1}{2} g_{\alpha\beta\,|\,\gamma} (= \frac{1}{2}\, \partial g_{\alpha\beta} / \partial y^{\gamma})$. In this case, from (2.6) and (2.10) we have

(2.16) $$C_{\alpha\beta\gamma} = C_{ijk} B^{ijk}_{\alpha\beta\gamma}, \quad C^{\alpha}_{\beta\gamma} = g^{\alpha\sigma} C_{\beta\sigma\gamma} = C^i_{jk} B^{\alpha jk}_{i\beta\gamma}.$$

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5) Here snd in the following, Latin indices a, b, c, ... run from $m+1$ to n.

6) We use this notation instead of $N_{ai} := g_{ij} N^j_a$ for the later summation convention.

7) If no confusion occurs, then we shall use the summation convention also for indices a, b, c, ...

Differentiating $g^{\alpha\sigma}g_{\sigma\varepsilon}=\delta^{\alpha}_{\varepsilon}$ by y^{γ} and contracting the result by $g^{\varepsilon\beta}$, from (2. 9), (2. 10), (2. 12) and (2. 16) we obtain

(2. 17) $$g^{\alpha\beta}{}_{\|\gamma}=-2g^{\alpha\sigma}C_{\sigma}{}^{\beta}{}_{\gamma}=-2g^{ih}C_{h}{}^{j}{}_{k}B^{\alpha\beta k}_{ij\gamma}=g^{ij}{}_{\|k}B^{\alpha\beta k}_{ij\gamma}.$$

We define six quantities as follows :

(2. 18) $$C_{a\beta\gamma}:=C_{ijk}N^{i}_{a}B^{jk}_{\beta\gamma},\ C_{ab\gamma}:=C_{ijk}N^{i}_{a}N^{j}_{b}B^{k}_{\gamma},\ C_{abc}:=C_{ijk}N^{i}_{a}N^{j}_{b}N^{k}_{c},$$
$$C_{a}{}^{\beta}{}_{\gamma}:=C_{i}{}^{j}{}_{k}N^{i}_{a}B^{\beta k}_{j\gamma},\ C_{a}{}^{b}{}_{\gamma}:=C_{i}{}^{j}{}_{k}N^{i}_{a}N^{b}_{j}B^{k}_{\gamma},\ C_{a}{}^{b}{}_{c}:=C_{i}{}^{j}{}_{k}N^{i}_{a}N^{b}_{j}N^{k}_{c}.$$

Then we can easily prove

Lemma 2.1. *The six quantities in* (2. 18) *are all symmetric in the lower indices. Further the following relations hold :*

(2. 19) $$C_{a}{}^{\beta}{}_{\gamma}=g^{\beta\alpha}C_{a\alpha\gamma},\ C_{a}{}^{b}{}_{\gamma}=C_{ab\gamma},\ C_{a}{}^{b}{}_{c}=C_{abc},$$
$$C_{a\beta\gamma}B^{\beta}{}_{j}=C_{aj\gamma}-C_{ab\gamma}N^{b}_{j},\ C_{a}{}^{\beta}{}_{\gamma}B^{j}_{\beta}=C_{a}{}^{j}{}_{\gamma}-C_{a}{}^{b}{}_{\gamma}N^{j}_{b},$$

where $C_{aj\gamma}=C_{ijk}N^{i}_{a}B^{k}_{\gamma}$ *and* $C_{a}{}^{j}{}_{\gamma}=C_{i}{}^{j}{}_{k}N^{i}_{a}B^{k}_{\gamma}$.

If we differentiate $B^{\alpha}{}_{i}$ in (2. 10) by y^{γ} and make use of (2. 10), (2. 17) and (2. 19), then we obtain

(2. 20) $$B^{\alpha}{}_{i\|\gamma}=2C_{b}{}^{\alpha}{}_{\gamma}N^{b}_{i}.$$

Differentiating the second in (2. 12) by y^{γ}, from (2. 13) and (2. 20) we have

(2. 21) $$N^{i}_{b\|\gamma}N^{b}_{j}+N^{i}_{b}N^{b}_{j\|\gamma}=-2C_{b}{}^{\alpha}{}_{\gamma}N^{b}_{j}B^{i}_{\alpha}.$$

If we contract (2. 21) by N^{a}_{i}, then we obtain

(2. 22) $$N^{a}_{j\|\gamma}=\lambda^{a}_{b\gamma}N^{b}_{j},\ \lambda^{a}_{b\gamma}:=-N^{a}_{i}N^{i}_{b\|\gamma}=N^{a}_{i\|\gamma}N^{i}_{b}.$$

Since $N^{a}_{j}=g_{jk}N^{k}_{a}$, from (2. 22) we have

(2. 23) $$N^{i}_{a\|\gamma}=\lambda^{a}_{b\gamma}N^{b}_{j}g^{ij}-2C_{a}{}^{i}{}_{\gamma}.$$

Applying (2. 23) to the second in (2. 22), from Lemma 2. 1 we get

(2. 24). $$\lambda^{a}_{b\gamma}+\lambda^{b}_{a\gamma}=2C_{b}{}^{a}{}_{\gamma}=2C_{a}{}^{b}{}_{\gamma}.$$

If we apply (2. 24) to (2. 23) and take account of (2. 19), then we obtain

(2.25) $$N^{i}_{a|\gamma} = -2C^{\ \beta}_{a\gamma} B^{i}_{\beta} - \lambda^{b}_{a\gamma} N^{i}_{b}.$$

For the later use, we define two tensors on M_m as follows :

(2.26) $$\nu_{ba\beta\gamma} = C_{ba\beta\|\gamma} + 2C_{a\beta\sigma} C^{\ \sigma}_{b\gamma} + C_{ca\beta} \lambda^{c}_{b\gamma},$$

(2.27) $$\mu^{\ a}_{b\beta\gamma} := C^{\ a}_{b\beta|\gamma} + C^{\ a}_{c\beta} \lambda^{c}_{b\gamma}.$$

Then we can state

Lemma 2.2. *The tensor $\nu_{ba\beta\gamma}$ is symmetric in all Greek indices, while the tensor $\mu^{\ a}_{b\beta\gamma}$ is symmetric in β and γ.*

Proof. From (2.18), (2.24), (2.25) and (2.26) it follows that

$$B^{ijk}_{a\beta\gamma} C_{ijk\|h} N^{h}_{b} = B^{ijk}_{a\beta\gamma} C_{ijh\|k} N^{h}_{b} = B^{ij}_{a\beta} C_{ijh|\gamma} N^{h}_{b}$$

$$= C_{ba\beta|\gamma} - B^{ij}_{a\beta} C_{ijh} N^{h}_{b|\gamma} = \nu_{ba\beta\gamma}.$$

which shows that the tensor $\nu_{ba\beta\gamma}$ is symmetric in α, β and γ. Because the tensor C_{ijk} is also symmetric in i, j and k.

Next, by virtue of (2.17), (2.26) and (2.27) we get

(2.28) $$\nu_{b\varepsilon\beta\gamma} g^{\varepsilon a} - 2(C^{\ a}_{\gamma\sigma} C^{\ \sigma}_{b\beta} + C^{\ a}_{\beta\sigma} C^{\ \sigma}_{b\gamma}) = \mu^{\ a}_{b\beta\gamma},$$

which indicates that the tensor $\mu^{\ a}_{a\beta\gamma}$ is symmetric in β and γ. Q. E. D.

From (2.28) we obtain

(2.29) $$\mu_{ba\beta\gamma} := g_{a\varepsilon} \mu^{\ \varepsilon}_{b\beta\gamma} = \nu_{ba\beta\gamma} - 2(C_{\gamma a\sigma} C^{\ \sigma}_{b\beta} + C_{\beta a\sigma} C^{\ \sigma}_{b\gamma}).$$

§3. Induced Matsumoto connections. On M_n the absolute differential of a vector y^i with respect to an $M\Gamma$ is given by

(3.1) $$Dy^{i} = dy^{i} + \Gamma^{i}_{\ k}(x^{j}, y^{j})\, dx^{k},$$

while the absolute differential of a vector $X^{i}(x^{j}, y^{j})$ is given by

$$DX^{i} = dX^{i} + (\Gamma^{i}_{jk} + \widetilde{C}^{\ i}_{jh} \Gamma^{h}_{\ k}) X^{j} dx^{k} + \widetilde{C}^{\ i}_{jk} X^{j} dy^{k}$$

(3.2) $$= X^{i}_{\ |k} dx^{k} + X^{i}|_{k} Dy^{k},$$

$$X^{i}_{\ |k} = \delta_{k} X^{i} - \Gamma^{i}_{jk} X^{j}, \quad X^{i}|_{k} = X^{i}_{\ |k} + \widetilde{C}^{\ i}_{jk} X^{j},$$

where $\delta_k X^i = \partial X^i / \partial x^k - \Gamma^j{}_k X^i{}_{\|j}$ and $X^i{}_{\|k} = \partial X^i / \partial y^k$.

Let vectors y^i and X^i be tangent to M_m, that is, they satisfy (2. 4) and (2. 3) respectively. Then the expressions (3. 1) and (3. 2) are written in

$$\text{(3. 3)} \qquad \begin{aligned} Dy^i &= (B^i_{0\gamma} + \Gamma^i{}_k B^k{}_\gamma) du^\gamma + B^i{}_\beta dy^\beta, \\ B^i_{\beta\gamma} &= \partial B^i{}_\beta / \partial u^\gamma, \ B^i_{0\gamma} = B^i_{\beta\gamma} y^\beta, \end{aligned}$$

$$\text{(3. 4)} \qquad \begin{aligned} DX^i = B^i{}_\beta dX^\beta &+ (B^i_{\beta\gamma} + \Gamma^i_{jk} B^{jk}_{\beta\gamma} + \widetilde{C}^i_{jh} \Gamma^h{}_k B^{jk}_{\beta\gamma} \\ &+ \widetilde{C}^i_{jk} B^k_{0\gamma} B^j{}_\beta) X^\beta du^\gamma + \widetilde{C}^i_{jk} B^{jk}_{\beta\gamma} X^\beta dy^\gamma. \end{aligned}$$

Now we define Dy^α and DX^α as follows :

$$\text{(3. 5)} \qquad Dy^\alpha = B^\alpha{}_i Dy^i, \ DX^\alpha = B^\alpha{}_i DX^i.$$

Then on M_m we can define a connection $\overline{\Gamma} = (\Gamma^\alpha_{\beta\gamma}, \ \Gamma^\alpha{}_\gamma, \ \widetilde{C}^\alpha_{\beta\gamma})$ by means of (3. 5). In terms of this connection, Dy^α and DX^α are expressible in

$$\text{(3. 6)} \qquad Dy^\alpha = dy^\alpha + \Gamma^\alpha{}_\gamma du^\gamma,$$

$$\text{(3. 7)} \qquad \begin{aligned} DX^\alpha &= dX^\alpha + (\Gamma^\alpha_{\beta\gamma} + \widetilde{C}^\alpha_{\beta\sigma} \Gamma^\sigma{}_\gamma) X^\beta du^\gamma + \widetilde{C}^\alpha_{\beta\gamma} X^\beta dy^\gamma \\ &= X^\alpha{}_{|\gamma} du^\gamma + X^\alpha|_\gamma Dy^\gamma, \\ X^\alpha{}_{|\gamma} &= \delta_\gamma X^\alpha + \Gamma^\alpha_{\beta\gamma} X^\beta, \ X^\alpha|_\gamma = X^\alpha{}_{\|\gamma} + \widetilde{C}^\alpha_{\beta\gamma} X^\beta, \end{aligned}$$

where $\delta_\gamma X^\alpha = \partial X^\alpha / \partial u^\gamma - \Gamma^\beta{}_\gamma X^\alpha{}_{\|\beta}$ and $X^\alpha{}_{\|\gamma} = \partial X^\alpha / \partial y^\gamma$.

If we substitute (3. 3) and (3. 4) into (3. 5) and compare the results with (3. 6) and (3. 7), then we have

$$\text{(3. 8)} \qquad \widetilde{C}^\alpha_{\beta\gamma} = \widetilde{C}^i_{jk} B^{\alpha jk}_{i\beta\gamma},$$

$$\text{(3. 9)} \qquad \Gamma^\alpha{}_\gamma = B^\alpha{}_i (B^i_{0\gamma} + \Gamma^i{}_k B^k{}_\gamma), \ B^i_{0\gamma} = y^\beta B^i_{\beta\gamma},$$

$$\text{(3. 10)} \qquad \Gamma^\alpha_{\beta\gamma} = B^\alpha{}_i (B^i_{\beta\gamma} + \Gamma^i_{jk} B^{jk}_{\beta\gamma}) - \widetilde{C}^\alpha_{\beta\sigma} \Gamma^\sigma{}_\gamma + B^\alpha{}_i \widetilde{C}^i_{jh} (\Gamma^h{}_k B^k{}_\gamma + B^h_{0\gamma}) B^j{}_\beta.$$

If we apply (3. 8) and (3. 9) to (3. 10) and use (2. 12) and (2. 13), then we obtain

$$\text{(3. 11)} \qquad \Gamma^\alpha_{\beta\gamma} = B^\alpha{}_i (B^i_{\beta\gamma} + \Gamma^i_{jk} B^{jk}_{\beta\gamma}) + \widetilde{C}^\alpha_{\beta b} H^b{}_\gamma,$$

where we put

$$(3.12)\qquad \widetilde{C}^{a}_{ab} = \widetilde{C}^{i}_{jk} B^{ja}_{ai} N^{k}_{b},$$

$$(3.13)\qquad H^{b}_{\gamma} = N^{b}_{i} (B^{i}_{0\gamma} + \Gamma^{i}_{k} B^{k}_{\gamma}).$$

We shall call a connection $\bar{\Gamma} = (\Gamma^{a}_{\beta\gamma}, \Gamma^{a}_{\gamma}, \widetilde{C}^{a}_{\beta\gamma})$ on M_m defined by (3. 8), (3. 9) and (3. 11) the *induced Matsumoto connection*, and denote it by $IM\Gamma$.

We put

$$(3.14)\qquad \widetilde{C}^{a}_{\beta b} = B^{j}_{\beta} N^{a}_{i} N^{k}_{b} \widetilde{C}^{i}_{jk},$$

$$(3.15)\qquad H^{a}_{\beta\gamma} = N^{a}_{i} (B^{i}_{\beta\gamma} + \Gamma^{i}_{jk} B^{jk}_{\beta\gamma}) + \widetilde{C}^{a}_{\beta b} H^{b}_{\gamma}.$$

Then if we contract (3. 11) by B^{i}_{a} and use (3. 15), then we have

$$(3.16)\qquad B^{i}_{\beta\gamma} + \Gamma^{i}_{jk} B^{jk}_{\beta\gamma} + \widetilde{C}^{i}_{jk} B^{j}_{\beta} N^{k}_{a} H^{a}_{\gamma} = B^{i}_{a} \Gamma^{a}_{\beta\gamma} + N^{i}_{a} H^{a}_{\beta\gamma}.$$

In view of (3. 16) we can define the h-relative covariant derivative of B^{i}_{β} with respect to $IM\Gamma$ as follows :

$$(3.17)\qquad H^{i}_{\beta\gamma} = B^{i}_{\beta|\gamma} = \partial B^{i}_{\beta} / \partial u^{\gamma} + \Gamma^{i}_{j\gamma} B^{j}_{\beta} - \Gamma^{a}_{\beta\gamma} B^{i}_{a},$$

$$(3.18)\qquad \Gamma^{i}_{j\gamma} = \Gamma^{i}_{jk} B^{k}_{\gamma} + \widetilde{C}^{i}_{jk} N^{k}_{a} H^{a}_{\gamma}.$$

It then follows from (3. 16) ~ (3. 18) that $H^{a}_{\beta\gamma} = N^{a}_{i} H^{i}_{\beta\gamma}$.

Let $X^{ia}_{j\beta}$ be an object defined on M_m such that it is a tensor in M_n of type (1, 1) and, at the same time, a tensor in M_m of type (1, 1). Then the relative h- and v-covariant derivatives of $X^{ia}_{j\beta}$ are defined as follows :

$$(3.19)\qquad X^{ia}_{j\beta|\gamma} = \delta_{\gamma} X^{ia}_{j\beta} + X^{ka}_{j\beta} \Gamma^{i}_{k\gamma} - X^{ia}_{k\beta} \Gamma^{k}_{j\gamma} + X^{i\sigma}_{j\beta} \Gamma^{a}_{\sigma\gamma} - X^{ia}_{j\sigma} \Gamma^{\sigma}_{\beta\gamma},$$

where δ_{γ} and $\Gamma^{i}_{j\gamma}$ are defined in (3. 7) and (3. 18) respectively.

$$(3.20)\qquad X^{ia}_{j\beta}|_{\gamma} = X^{ia}_{j\beta|\gamma} + X^{ka}_{j\beta} \widetilde{C}^{i}_{k\gamma} - X^{ia}_{k\beta} \widetilde{C}^{k}_{j\gamma} + X^{i\sigma}_{j\beta} \widetilde{C}^{a}_{\sigma\gamma} - X^{ia}_{j\sigma} \widetilde{C}^{\sigma}_{\beta\gamma},$$

where $\widetilde{C}^{i}_{k\gamma} = \widetilde{C}^{i}_{kj} B^{j}_{\gamma}$.

From (2. 12), (2. 13), (3. 16), (3. 17) and (3. 20) we obtain

$$(3.21)\qquad B^{i}_{\beta|\gamma} = H^{i}_{\beta\gamma} = N^{i}_{a} H^{a}_{\beta\gamma},\quad B^{i}_{\beta}|_{\gamma} = N^{i}_{a} \widetilde{C}^{a}_{\beta\gamma},$$

where $\widetilde{C}^{a}_{\beta\gamma} = N^{a}_{i}\widetilde{C}^{i}_{jk}B^{jk}_{\beta\gamma}$. Because of (2. 12), (2. 13), (3. 9) and (3. 13) we have

(3. 22) $$B^{i}_{0\gamma} - B^{i}_{\beta}\Gamma^{\beta}_{\gamma} = -\Gamma^{i}_{k}B^{k}_{\gamma} + N^{i}_{a}H^{a}_{\gamma},$$

which implies

(3 23) $$\delta_{\gamma} = B^{k}_{\gamma}\partial/\partial x^{k} + (B^{i}_{0\gamma} - B^{i}_{\beta}\Gamma^{\beta}_{\gamma})\partial/\partial y^{i} = B^{i}_{\gamma}\delta_{i} + H^{a}_{\gamma}N^{i}_{a}\partial/\partial y^{i},$$

where $\delta_{i} = \partial/\partial x^{i} - \Gamma^{j}_{i}\partial/\partial y^{j}$.

Then it easily follows from (3. 19), (3. 20) and (2. 23) that

(3. 24) $$g_{ij|\gamma} = g_{ij|k}B^{k}_{\gamma} + g_{ij}|_{k}N^{k}_{a}H^{a}_{\gamma},\quad g_{ij}|_{\gamma} = g_{ij}|_{k}B^{k}_{\gamma}.$$

Further from (2. 14) and (3. 23) we have

(3. 25) $$\overline{L}_{|\gamma} = B^{i}_{\gamma}L_{|i}.$$

From (3. 25) we can state

Lemma 3.1. *If an MΓ is metrical, then the IMΓ is also metrical.*

From (2. 11), (3. 20), (3. 21) and (3. 24) we obtain

(3. 26) $$g_{\alpha\beta|\gamma} = g_{ij|k}B^{ijk}_{\alpha\beta\gamma} + g_{ij}|_{k}B^{ij}_{\alpha\beta}N^{k}_{a}H^{a}_{\gamma},$$

(3. 27) $$g_{\alpha\beta}|_{\gamma} = g_{ij}|_{k}B^{ijk}_{\alpha\beta\gamma}.$$

Because of (3. 26) and (3. 27) we can state

Lemma 3.2. *If an MΓ is v-metrical, then the IMΓ is also v-metrical. If an MΓ is both h-metrical and v-metrical, then the IMΓ is h-metrical.*

The deflexion tensor $\widetilde{D}^{\alpha}_{\gamma}$ with respect to *IMΓ* is, because of (3. 9) and (3. 10), given by

(3. 28) $$\widetilde{D}^{\alpha}_{\gamma} = \Gamma^{\alpha}_{0\gamma} - \Gamma^{\alpha}_{\gamma} = D^{i}_{k}B^{\alpha k}_{i\gamma} + \widetilde{C}^{\alpha}_{0b}H^{b}_{\gamma},$$

where D^{i}_{k} is the deflexion tensor with respect to *MΓ* and $\widetilde{C}^{a}_{0b} = y^{\beta}\widetilde{C}^{a}_{\beta b}$.

We shall say that the *IMΓ* is *dft-natural* if the deflexion tensor is given by $\widetilde{D}^{\alpha}_{\gamma} = D^{i}_{k}B^{\alpha k}_{i\gamma}$. Then from (3. 28) we can state

Lemma 3.3. *The IMΓ is dft-natural if and only if the following equation holds:*

(3. 29) $$\widetilde{C}^{a}_{0b}H^{b}_{\gamma} = 0.$$

Note 3.1. We shall say that an $M\Gamma$ is *dft-free* if it satisfies (F2). In this case, the $IM\Gamma$ is also *dft-free* if and only if it is *dft-natural.*

From (3. 26) and (3. 27) we have

(3. 30) $$Dg_{\alpha\beta}=g_{ij|k}B^{ijk}_{\alpha\beta\gamma}du^{\gamma}+g_{ij}|_{k}B^{ij}_{\alpha\beta}(N^{k}_{b}H^{b}_{\gamma}du^{\gamma}+B^{k}_{\gamma}Dy^{\gamma}).$$

We shall say that an $M\Gamma$ is *Dy-reciprocal* if it satisfies (F4). Then it is easily seen that an $M\Gamma$ is *Dy*-reciprocal if and only if

(3. 31) $$y^{i}g_{ij|k}=0,\ y^{i}g_{ij}|_{k}=0.$$

Because of (3. 30) and (3. 31) we can state

Lemma 3.4. *If an $M\Gamma$ is Dy-reciprocal, then the $IM\Gamma$ is also Dy-reciprocal.*

Let $\gamma^{\gamma}_{\alpha\beta}$ and γ^{k}_{ij} be the christoffel symbols of the second kind formed with $g_{\alpha\beta}$ and g_{ij} respectively. Then they are related by

(3. 32) $$\gamma^{\gamma}_{\alpha\beta}=B^{\gamma}_{i}(B^{i}_{\alpha\beta}+\gamma^{i}_{jk}B^{jk}_{\alpha\beta})+g^{\gamma\sigma}C_{ijk}(B^{ij}_{\beta\sigma}B^{k}_{0\alpha}+B^{ij}_{\sigma\alpha}B^{k}_{0\beta}-B^{ij}_{\alpha\beta}B^{k}_{0\gamma}).$$

Contracting (3. 32) by $y^{\alpha}y^{\beta}$, we have

(3. 33) $$2G^{\gamma}=\gamma^{\gamma}_{\alpha\beta}y^{\alpha}y^{\beta}=B^{\gamma}_{i}(B^{i}_{00}+2G^{i}).$$

On the other hand, contraction of (3. 9) by y^{γ}, because of (3. 33), yields

(3. 34) $$\Gamma^{a}_{\gamma}y^{\gamma}=2G^{a}+T^{i}{}_{0}B^{a}_{i}.$$

An $M\Gamma$ is called a *geo-path connection* if it satisfies (F5). Then it follows from (1. 2) that an $M\Gamma$ is a geo-path connection if and only if $T^{i}{}_{0}=0$. Therefore from (3. 34) we can state

Lemma 3.5. *If an $M\Gamma$ is a geo-path connection on M_n, then the $IM\Gamma$ is also a geo-path connection on M_m.*

We put

(3. 36) $$T^{a}_{\gamma}=T^{i}{}_{k}B^{ak}_{i\gamma},\ Q^{a}_{\beta\gamma}=Q^{i}_{jk}B^{ajk}_{i\beta\gamma},$$

(3. 37) $$\overset{b}{\Gamma}{}^{a}_{\beta\gamma}=B^{a}_{i}(B^{i}_{\beta\gamma}+G^{i}_{jk}B^{jk}_{\beta\gamma}),\ \overset{b}{\Gamma}{}^{a}_{\gamma}=B^{a}_{i}(B^{i}_{0\gamma}+G^{i}{}_{k}B^{k}_{\gamma}),$$

(3. 38) $$\overset{b}{H}{}^{a}_{\gamma}=N^{a}_{i}(B^{i}_{0\gamma}+G^{i}{}_{k}B^{k}_{\gamma}),\ \overset{b}{H}{}^{a}_{0}=N^{a}_{i}(B^{i}_{00}+2G^{i}).$$

Differentiating (3. 33) by y^{γ} and using (2. 20) and (3. 37), we have

(3. 39) $\quad G^{\alpha}_{\ \gamma}=G^{\alpha}_{\ \|\gamma}=\overset{b}{\Gamma}{}^{\alpha}_{\ \gamma}+C_{\gamma\ b}^{\ \alpha}\overset{b}{H}{}^{b}_{\ 0}.$

Subtracting (3. 9) from (3. 38) and using (3. 36), we get

(3. 40) $\quad \widetilde{T}^{\alpha}_{\ \gamma}=\Gamma^{\alpha}_{\ \gamma}-G^{\alpha}_{\ \gamma}=T^{\alpha}_{\ \gamma}-C_{\gamma\ b}^{\ \alpha}\overset{b}{H}{}^{b}_{\ 0}.$

Differentiating (3. 9) by y^{β} and using (2. 20), we have

(3. 41) $\quad \Gamma^{\alpha}_{\ \gamma\|\beta}=B^{\alpha}_{\ i}(B^{\ i}_{\beta\gamma}+\Gamma^{i}_{\ k\|j}B^{kj}_{\gamma\beta})+2C_{\beta\ b}^{\ \alpha}H^{\ b}_{\gamma}.$

Since $\Gamma^{i}_{\ k\|j}=\Gamma^{\ i}_{j\ k}-Q^{\ i}_{j\ k}$, from (3. 11), (3. 36) and (3. 41) we obtain

(3. 42) $\quad \widetilde{Q}^{\ \alpha}_{\beta\gamma}=\Gamma^{\ \alpha}_{\beta\gamma}-\Gamma^{\alpha}_{\ \gamma\|\beta}=Q^{\ \alpha}_{\beta\gamma}+(\widetilde{C}^{\ \alpha}_{\beta b}-2C^{\ \alpha}_{\beta b})H^{\ b}_{\gamma}.$

Consequently we can state

Theorem 3.1. *The IMΓ is a Matsumoto connection on* M_m *determined by three tensors* $\widetilde{T}^{\alpha}_{\ \gamma}$, $\widetilde{Q}^{\ \alpha}_{\beta\gamma}$ *and* $\widetilde{C}^{\ \alpha}_{\beta\gamma}$ *in* (3. 40), (3. 42) *and* (3. 8).

The original connection MΓ is a Matsumoto connection determined by three tensors $T^{i}_{\ k}$, $Q^{\ i}_{j\ k}$ and $\widetilde{C}^{\ i}_{j\ k}$. The tensors induced on M_m from the above tensors on M_m are $T^{\alpha}_{\ \gamma}$, $Q^{\ \alpha}_{\beta\gamma}$ and $\widetilde{C}^{\ \alpha}_{\beta\gamma}$. Therefore the *intrinsic* MΓ*) on M_m is defined as the Matsumoto connection on M_m determined by $T^{\alpha}_{\ \gamma}$, $Q^{\ \alpha}_{\beta\gamma}$ and $\widetilde{C}^{\ \alpha}_{\beta\gamma}$. If we denote this connection by $\overset{m}{\Gamma}=(\overset{m}{\Gamma}{}^{\ \alpha}_{\beta\gamma},\ \overset{m}{\Gamma}{}^{\alpha}_{\ \gamma},\ \widetilde{C}^{\ \alpha}_{\beta\gamma})$, then from (3. 40) ~(3. 42) we obtain

(3. 43)$_1$ $\quad \overset{m}{\Gamma}{}^{\alpha}_{\ \gamma}=\Gamma^{\alpha}_{\ \gamma}+C_{\gamma\ b}^{\ \alpha}\overset{b}{H}{}^{b}_{\ 0},$

(3. 43)$_2$ $\quad \overset{m}{\Gamma}{}^{\ \alpha}_{\beta\gamma}=\Gamma^{\ \alpha}_{\beta\gamma}+(C_{\gamma\ b}^{\ \alpha}\overset{b}{H}{}^{b}_{\ 0})_{\|\beta}-(\widetilde{C}^{\ \alpha}_{\beta b}-2C^{\ \alpha}_{\beta b})H^{\ b}_{\gamma}.$

Consequently, because of (3. 43)$_1$ and (3. 43)$_2$ we can state

Theorem 3.2. *The IMΓ is the intrinsic MΓ on* M_m *if and only if the following equation hold*:

(3. 43)$_3$ $\quad C_{\gamma\ b}^{\ \alpha}\overset{b}{H}{}^{b}_{\ 0}=0,\ (\widetilde{C}^{\ \alpha}_{\beta b}-2C^{\ \alpha}_{\beta b})H^{\ b}_{\gamma}=0.$

From (2. 11) we have

(3. 44) $\quad g_{ij}B^{i}_{\ \alpha}N^{\ i}_{a}=0.$

Differentiating (3. 44) h-covariantly by u^{γ} and using (2. 10), (2. 11) and (3. 21), we obtain

*) This connection cannot be defined in the usual sense. So we define it as above.

(3.45) $g_{\alpha\beta} B^{\beta}_{j} N^{j}_{a|\gamma} + \delta_{ab} H^{\ b}_{\alpha\gamma} + g_{ij|\gamma} B^{i}_{\alpha} N^{j}_{a} = 0.$

If we contract (3.45) by $g^{\alpha\varepsilon}$ and successibly contract the result by B^{i}_{ε}, then we have

(3.46) $N^{i}_{a|\gamma} = (N^{b}_{j} N^{j}_{a|\gamma}) N^{i}_{b} - g^{\alpha\varepsilon} (g_{jk|\gamma} B^{j}_{\alpha} N^{k}_{a} + \delta_{ab} H^{\ b}_{\alpha\gamma}) B^{i}_{\varepsilon}.$

Similarly, by differentiating (3.44) v-covariantly by y^{γ}, we obtain

(3.47) $N^{i}_{a}|_{\gamma} = (N^{b}_{j} N^{j}_{a}|_{\gamma}) N^{i}_{b} - g^{\alpha\varepsilon} (g_{jk}|_{\gamma} B^{j}_{\alpha} N^{k}_{a} + \delta_{ab} \widetilde{C}^{\ b}_{\alpha\beta}) B^{i}_{\varepsilon}.$

Contracting (3.5) by B^{j}_{α}, from (2.12) we have

(3.48) $Dy^{j} = B^{j}_{\alpha} Dy^{\alpha} + N^{j}_{b} (N^{b}_{i} Dy^{i}), \quad DX^{i} = B^{j}_{\alpha} DX^{\alpha} + N^{j}_{b} (N^{b}_{i} DX^{i}).$

If we contract the first in (3.3) by N^{a}_{i} and use (3.13), then we obtain $N^{a}_{i} Dy^{i} = H^{a}_{\gamma} du^{\gamma}$. Therefore from the first in (3.48) we have

(3.49) $Dy^{i} = B^{i}_{\gamma} Dy^{\gamma} + N^{i}_{a} (H^{a}_{\gamma} du^{\gamma}).$

Contracting (3.4) by N^{a}_{i} and making use of (3.14), (3.15) and (3.23), we obtain $N^{a}_{i} DX^{i} = H^{\beta}_{a} (H^{\ a}_{\beta\gamma} du^{\gamma} + \widetilde{C}^{\ a}_{\beta\gamma} Dy^{\gamma})$. Hence because of the second in (3.48), DX^{i} is expressible in

(3.50) $DX^{i} = B^{i}_{\gamma} DX^{\gamma} + N^{i}_{a} X^{\beta} (H^{\ a}_{\beta\gamma} du^{\gamma} + \widetilde{C}^{\ a}_{\beta\gamma} Dy^{\gamma}).$

The vector H^{a}_{γ} will be called the *normal curvature vector* in a normal direction N^{i}_{a}, while the tensors $H^{\ a}_{\beta\gamma}$ and $\widetilde{C}^{\ a}_{\beta\gamma}$ will be called the *second fundamental h- and v-tensors* in the same direction N^{i}_{a} respectively.

For the later use, we shall give

Lemma 3.6. *The following facts are mutually equivalent:*

(1) *The non-linear connection Γ^{α}_{γ} of IMΓ is the intrinsic one.*

(2) $C^{\ a}_{\gamma b} \overset{b}{H}{}^{b}_{0} = 0.$ (3) $2C^{\ a}_{\gamma b} \overset{b}{H}{}^{b}_{\beta} + \mu^{\ a}_{b\beta\gamma} \overset{b}{H}{}^{b}_{0} = 0.$

(4) $C^{\ a}_{\beta b} \overset{b}{H}{}^{b}_{\gamma} = C^{\ a}_{\gamma b} \overset{b}{H}{}^{b}_{\beta}.$

Proof. It is easily seen from $(3.43)_1$ that (1) and (2) are mutually equivalent. If we differentiate (2) by y^{γ}, then from (2.22), (2.26) and (3.38) we have (3). Conversely, if we contract (3) by y^{β}, then we obtain (2). From (3) and Lemma 2.2 it follows that (4) holds. Conversely, if we contract (4) by y^{β}, then we get (2) and hence (3). Q.E.D.

§4. Various tensors on M_m and the generalized Gauss-Codazzi equations. With respect to the $IM\Gamma=(\Gamma^{\alpha}_{\beta\gamma}, \Gamma^{\alpha}_{\gamma}, \widetilde{C}^{\alpha}_{\beta\gamma})$, we can define the torsion tensors, the curvature tensors and the mixed tensors.

Firstly, the h-torsion tensor $\widetilde{\tau}^{\alpha}_{\beta\gamma}$, the hv-torsion tensor $\widetilde{P}^{\alpha}{}_{\beta\gamma}$ and v-torsion tensor $\widetilde{S}^{\alpha}_{\beta\gamma}$ are defined by

(4.1) $$\widetilde{\tau}^{\alpha}_{\beta\gamma}=\Gamma^{\alpha}_{\beta\gamma}-\Gamma^{\alpha}_{\gamma\beta},\quad \widetilde{P}^{\alpha}{}_{\beta\gamma}=\Gamma^{\alpha}{}_{\beta\|\gamma}-\Gamma^{\alpha}_{\gamma\beta},\quad \widetilde{S}^{\alpha}_{\beta\gamma}=\widetilde{C}^{\alpha}_{\beta\gamma}-\widetilde{C}^{\alpha}_{\gamma\beta}.$$

The curvature tensor $\widetilde{R}^{\alpha}{}_{\beta\gamma}$ with respect to the induced non-linear connection $\Gamma^{\alpha}{}_{\gamma}$ is defined by

(4.2) $$\widetilde{R}^{\alpha}{}_{\beta\gamma}=\delta_{\gamma}\Gamma^{\alpha}{}_{\beta}-\delta_{\beta}\Gamma^{\alpha}{}_{\gamma},$$

which is called also the $h(h)$-torsion tensor.

Now we consider a scalar $X(x^i, y^i)$ on M_n, which is also a scalar $X(x^i(u^{\alpha}), B^i{}_{\alpha} y^{\alpha})$ on M_m. Then for the X, the following Ricci's identities hold :

(4.3)
$$\text{(a)}\quad X_{|\beta|\gamma}-X_{|\gamma|\beta}=-X_{|\alpha}\widetilde{\tau}^{\alpha}_{\beta\gamma}-X|_{\alpha}\widetilde{R}^{\alpha}{}_{\beta\gamma},$$
$$\text{(b)}\quad X_{|\beta}|_{\gamma}-X|_{\gamma|\beta}=-X_{|\alpha}\widetilde{C}^{\alpha}_{\beta\gamma}-X|_{\alpha}\widetilde{P}^{\alpha}{}_{\beta\gamma},$$
$$\text{(c)}\quad X|_{\beta}|_{\gamma}-X|_{\gamma}|_{\beta}=-X|_{\alpha}\widetilde{S}^{\alpha}_{\beta\gamma}.$$

Because of (3. 23) we have $X|_{\alpha}=B^i{}_{\alpha}X|_i$ and $X_{|\alpha}=B^i{}_{\alpha}X_{|i}+N^i_b H^b_{\alpha}X|_i$. Therefore it follows from (3. 18)~(3. 21), (3. 23) and the Ricci's identities on M_n that both sides of (4. 3) are expressible in linear homogeneous expressions of $X_{|i}$ and $X|_i$. By comparing the coefficients of them we obtain

(4.4)
$$\text{(a)}_1\quad B^i_{\epsilon}\widetilde{R}^{\epsilon}{}_{\beta\gamma}+N^i_{\alpha}H^{\alpha}_{\epsilon}\widetilde{\tau}^{\epsilon}_{\beta\gamma}=\widetilde{R}^i{}_{jk}B^{jk}_{\beta\gamma}+\widetilde{P}^i{}_{jk}(B^j{}_{\beta}H^{\alpha}_{\gamma}-B^j{}_{\gamma}H^{\alpha}_{\beta})N^k_{\alpha}$$
$$+N^j_{\alpha}N^k_b H^{\alpha}_{\beta}H^b_{\gamma}\widetilde{S}^i_{jk}-(H^{\alpha}_{\beta|\gamma}N^i_{\alpha}+H^{\alpha}_{\beta}N^i_{\alpha|\gamma}-\beta\,|\gamma),$$
$$\text{(a)}_2\quad B^i_{\epsilon}\widetilde{\tau}^{\epsilon}_{\beta\gamma}=\tau^i_{jk}B^{jk}_{\beta\gamma}+\widetilde{C}^i_{jk}(B^j_{\beta}H^{\alpha}_{\gamma}-B^j_{\gamma}H^{\alpha}_{\beta})N^k_{\alpha}-N^i_{\alpha}(H^{\alpha}_{\beta\gamma}-H^{\alpha}_{\gamma\beta}),$$
$$\text{(b)}\quad B^i_{\epsilon}\widetilde{P}^{\epsilon}_{\beta\gamma}+N^i_{\alpha}H^{\alpha}_{\epsilon}\widetilde{C}^{\epsilon}_{\beta\gamma}=\widetilde{P}^i_{jk}B^{jk}_{\beta\gamma}+\widetilde{S}^i_{jk}N^j_{\alpha}H^{\alpha}_{\beta}B^k_{\gamma}-N^i_{\alpha}|_{\gamma}H^{\alpha}_{\beta}-(H^{\alpha}_{\beta}|_{\gamma}-H^{\alpha}_{\gamma\beta})N^i_{\alpha},$$
$$B^i{}_{\epsilon}\widetilde{C}^{\epsilon}_{\beta\gamma}=\widetilde{C}^i_{jk}B^{jk}_{\beta\gamma}-\widetilde{C}^{\alpha}_{\beta\gamma}N^i{}_{\alpha},$$
$$\text{(c)}\quad B^i_{\epsilon}\widetilde{S}^{\epsilon}_{\beta\gamma}=\widetilde{S}^i_{jk}B^{jk}_{\beta\gamma}-(\widetilde{C}^{\alpha}_{\beta\gamma}-\widetilde{C}^{\alpha}_{\gamma\beta})N^i{}_{\alpha},$$

where $\widetilde{R}^{\ i}_{jk}=\delta_k\Gamma'^{i}_{\ j}-\delta_j\Gamma'^{i}_{\ k}$, $\widetilde{P}^{\ i}_{jk}=\Gamma'^{i}_{\ j\|k}-\Gamma'^{i}_{kj}$, $\widetilde{C}^{\ i}_{j\ k}=\widetilde{C}^{\ i}_{j\ k}-\widetilde{C}^{\ i}_{k\ j}$, and the symbol $\beta\,|\gamma$ means the interchange of indices β and γ in the foregoing terms.

Note 4.1. $\widetilde{R}^{\ i}_{jk}$ is the curvature tensor on M_n with respect to $\Gamma'^{i}_{\ k}$ (or $h(h)$-torsion tensor), while $\widetilde{P}^{\ i}_{jk}$ and $\widetilde{S}^{\ i}_{jk}$ are the hv- and v-torsion tensors on M_n respectively.

Secondly, the h-curvature tensor $\widetilde{R}^{\ \alpha}_{\beta\gamma\sigma}$, the hv-curvature tensor $\widetilde{P}^{\alpha}_{\beta\gamma\sigma}$ and the v-curvature tensor $\widetilde{S}^{\ \alpha}_{\beta\gamma\sigma}$ are defined by

$$
(4.5)\quad
\begin{aligned}
&\text{(a)}\quad \widetilde{R}^{\ \alpha}_{\beta\gamma\sigma}=(\delta_\sigma\Gamma^{\alpha}_{\beta\gamma}+\Gamma^{\varepsilon}_{\beta\gamma}\Gamma^{\alpha}_{\varepsilon\sigma}-\gamma\,|\delta)+\widetilde{C}^{\ \alpha}_{\beta\varepsilon}\widetilde{R}^{\ \varepsilon}_{\gamma\sigma},\\
&\text{(b)}\quad \widetilde{P}^{\alpha}_{\beta\gamma\sigma}=\Gamma^{\alpha}_{\beta\gamma|\sigma}-\widetilde{C}^{\ \alpha}_{\beta\sigma|\gamma}+\widetilde{C}^{\ \alpha}_{\beta\varepsilon}\widetilde{P}^{\varepsilon}_{\gamma\sigma},\\
&\text{(c)}\quad \widetilde{S}^{\alpha}_{\beta\gamma\sigma}=\widetilde{C}^{\ \alpha}_{\beta\gamma|\sigma}+\widetilde{C}^{\ \varepsilon}_{\beta\gamma}\widetilde{C}^{\alpha}_{\varepsilon\sigma}-\gamma\,|\delta.
\end{aligned}
$$

Lastly, the mixed h-curvature tensor $\widetilde{R}^{\ i}_{j\beta\gamma}$, the mixed hv-curvature tensor $\widetilde{P}^{i}_{j\beta\gamma}$ and the mixed v-curvature tensor $\widetilde{S}^{\ i}_{j\beta\gamma}$ are defined by

$$
(4.6)\quad
\begin{aligned}
&\text{(a)}\quad \widetilde{R}^{\ i}_{j\beta\gamma}=[\delta_\gamma\Gamma^{\ i}_{j\beta}+\Gamma^{\ k}_{j\beta}\Gamma^{\ i}_{k\gamma}-\beta\,|\gamma]+\widetilde{C}^{\ i}_{j\varepsilon}\widetilde{R}^{\varepsilon}_{\ \beta\gamma},\\
&\text{(b)}\quad \widetilde{P}^{i}_{j\beta\gamma}=\Gamma^{\ i}_{j\beta\|\gamma}-\widetilde{C}^{\ i}_{j\gamma|\beta}+\widetilde{C}^{\ i}_{j\varepsilon}\widetilde{P}^{\varepsilon}_{\ \beta\gamma},\\
&\text{(c)}\quad \widetilde{S}^{\ i}_{j\beta\gamma}=\widetilde{C}^{\ i}_{j\beta\|\gamma}+\widetilde{C}^{\ k}_{j\beta}\widetilde{C}^{\ i}_{k\gamma}-\beta\,|\gamma,
\end{aligned}
$$

where $\Gamma^{\ i}_{j\beta}$ and $\widetilde{C}^{\ i}_{j\beta}$ are defined by (3. 18) and (3. 20).

Then for a mixed tensor $X^{\ i}_{\alpha}$, the following Ricci's identities hold :

$$
(4.7)\quad
\begin{aligned}
&\text{(a)}\quad X^{\ i}_{\alpha|\beta|\gamma}-\beta|\gamma=X^{\ j}_{\alpha}\widetilde{R}^{\ i}_{j\beta\gamma}-X^{\ i}_{\varepsilon}\widetilde{R}^{\ \varepsilon}_{\alpha\beta\gamma}-X^{\ i}_{\alpha|\varepsilon}\widetilde{\tau}^{\varepsilon}_{\beta\gamma}-X^{\ i}_{\alpha}|_{\varepsilon}\widetilde{R}^{\varepsilon}_{\ \beta\gamma},\\
&\text{(b)}\quad X^{\ i}_{\alpha|\beta}|_{\gamma}-X^{\ i}_{\alpha}|_{\gamma|\beta}=X^{\ j}_{\alpha}\widetilde{P}^{i}_{j\beta\gamma}-X^{\ i}_{\varepsilon}\widetilde{P}^{\varepsilon}_{\alpha\beta\gamma}-X^{\ i}_{\alpha|\varepsilon}\widetilde{C}^{\ \varepsilon}_{\beta\gamma}-X^{\ i}_{\alpha}|_{\varepsilon}\widetilde{P}^{\varepsilon}_{\ \beta\gamma},\\
&\text{(c)}\quad X^{\ i}_{\alpha}|_{\beta}|_{\gamma}-\beta\,|\gamma=X^{\ j}_{\alpha}\widetilde{S}^{\ i}_{j\beta\gamma}-X^{\ i}_{\varepsilon}\widetilde{S}^{\ \varepsilon}_{\alpha\beta\gamma}-X^{\ i}_{\alpha}|_{\varepsilon}\widetilde{S}^{\ \varepsilon}_{\beta\gamma}.
\end{aligned}
$$

Applying (a) ~(c) in (4. 7) to $B^{i}_{\ \alpha}$ and using (3. 21), we obtain

$$
(4.8)\quad
\begin{aligned}
&\text{(a)}\quad B^{i}_{\alpha|\beta|\gamma}-\beta\,|\gamma=B^{j}_{\ \alpha}\widetilde{R}^{\ i}_{j\beta\gamma}-B^{i}_{\ \varepsilon}\widetilde{R}^{\ \varepsilon}_{\alpha\beta\gamma}-H^{\ b}_{\alpha\varepsilon}\widetilde{\tau}^{\ \varepsilon}_{\beta\gamma}N^{\ i}_{b}-\widetilde{C}^{\ b}_{\alpha\varepsilon}\widetilde{R}^{\varepsilon}_{\ \beta\gamma}N^{\ i}_{b},\\
&\text{(b)}\quad B^{i}_{\alpha|\beta}|_{\gamma}-B^{i}_{\alpha}|_{\gamma|\beta}=B^{j}_{\ \alpha}\widetilde{P}^{i}_{j\beta\gamma}-B^{i}_{\ \varepsilon}\widetilde{R}^{\ \varepsilon}_{\alpha\beta\gamma}-H^{\ b}_{\alpha\varepsilon}\widetilde{C}^{\ \varepsilon}_{\beta r}N^{\ i}_{b}-\widetilde{C}^{\ b}_{\alpha\varepsilon}\widetilde{P}^{\varepsilon}_{\ \beta\gamma}N^{\ i}_{b},\\
&\text{(c)}\quad B^{i}_{\alpha}|_{\beta}|_{\gamma}-\beta\,|\gamma=B^{j}_{\ \alpha}\widetilde{S}^{\ i}_{j\beta\gamma}-B^{i}_{\ \varepsilon}\widetilde{S}^{\ \varepsilon}_{\alpha\beta\gamma}-\widetilde{C}^{\ b}_{\alpha\varepsilon}\widetilde{S}^{\ \varepsilon}_{\beta\gamma}N^{\ i}_{b}.
\end{aligned}
$$

If we contract (a)~(c) in (4. 8) by $B^{\sigma}_{\ i}$ and use (3. 21), (3. 46), (3. 47), then we get

(4.9)

(a) $\widetilde{R}_{\alpha\beta\gamma}^{\sigma}=\widetilde{R}_{j\beta\gamma}^{i}B_{\alpha i}^{j\sigma}+[H_{\alpha\beta}^{b}(g_{jk|\gamma}B_{\varepsilon}^{j}N_{b}^{k}+\delta_{bc}H_{\varepsilon\gamma}^{c})g^{\sigma\varepsilon}-\beta|\gamma]$,

(b) $\widetilde{P}_{\alpha\beta\gamma}^{\sigma}=\widetilde{P}_{j\beta\gamma}^{i}B_{\alpha i}^{j\sigma}+g^{\sigma\varepsilon}\{H_{\alpha\beta}^{b}(g_{jk}|_{\gamma}B_{\varepsilon}^{j}N_{b}^{k}+\delta_{bc}\widetilde{C}_{\varepsilon\gamma}^{c})$
$-\widetilde{C}_{\alpha\gamma}^{b}(g_{jk|\beta}B_{\varepsilon}^{j}N_{b}^{k}+\delta_{bc}H_{\varepsilon\beta}^{c})\}$,

(c) $\widetilde{S}_{\alpha\beta\gamma}^{\sigma}=\widetilde{S}_{j\beta\gamma}^{i}B_{\alpha i}^{j\sigma}+[\widetilde{C}_{\alpha\beta}^{b}(g_{jk}|_{\gamma}B_{\varepsilon}^{j}N_{b}^{k}+\delta_{bc}\widetilde{C}_{\varepsilon\gamma}^{c})g^{\sigma\varepsilon}-\beta|\gamma]$.

Similarly, by contracting (4.8) by N_i^c we obtain

(4.10)

(a) $\widetilde{R}_{j\beta\gamma}^{i}B_{\alpha}^{j}N_{i}^{c}=H_{\alpha\varepsilon}^{c}\tau_{\beta\gamma}^{\varepsilon}+\widetilde{C}_{\alpha\varepsilon}^{c}\widetilde{R}_{\beta\gamma}^{\varepsilon}+[H_{\alpha\beta|\gamma}^{c}+N_{b|\gamma}^{j}N_{j}^{c}H_{\alpha\beta}^{b}-\beta|\gamma]$,

(b) $\widetilde{P}_{j\beta\gamma}^{i}B_{\alpha}^{j}N_{i}^{c}=H_{\alpha\varepsilon}^{c}\widetilde{C}_{\beta\gamma}^{\varepsilon}+\widetilde{C}_{\alpha\varepsilon}^{c}\widetilde{P}_{\beta\gamma}^{\varepsilon}+N_{b}^{j}|_{\gamma}N_{j}^{c}H_{\alpha\beta}^{b}$
$+H_{\alpha\beta}^{c}|_{\gamma}-N_{b|\beta}^{j}N_{j}^{c}\widetilde{C}_{\alpha\gamma}^{b}-\widetilde{C}_{\alpha\gamma|\beta}^{c}$,

(c) $\widetilde{S}_{j\beta\gamma}^{i}B_{\alpha}^{j}N_{i}^{c}=\widetilde{C}_{\alpha\varepsilon}^{c}\widetilde{S}_{\beta\gamma}^{\varepsilon}+[N_{b}^{j}|_{\gamma}N_{j}^{c}\widetilde{C}_{\alpha\beta}^{b}+\widetilde{C}_{\alpha\beta}^{c}|_{\gamma}-\beta|\gamma]$.

The equations (4.9) and (4.10) are the generalized Gauss and Codazzi equations respectively.

If we calculate (4.6) and use (4.4), then we have

(4.11)

(a) $\widetilde{R}_{j\beta\gamma}^{i}=\widetilde{R}_{jkh}^{i}B_{\beta\gamma}^{kh}+\widetilde{P}_{jkh}^{i}(B_{\beta}^{k}H_{\gamma}^{b}-B_{\gamma}^{k}H_{\beta}^{b})N_{b}^{h}+\widetilde{S}_{jkh}^{i}N_{b}^{k}N_{c}^{h}H_{\beta}^{b}H_{\gamma}^{c}$,

(b) $\widetilde{P}_{j\beta\gamma}^{i}=\widetilde{P}_{jkh}^{i}B_{\beta\gamma}^{kh}+\widetilde{S}_{jkh}^{i}N_{b}^{k}H_{\beta}^{b}B_{\gamma}^{h}$,

(c) $\widetilde{S}_{j\beta\gamma}^{i}=\widetilde{S}_{jkh}^{i}B_{\beta\gamma}^{kh}$,

where $\widetilde{R}_{jkh}^{i}=[\delta_{h}\Gamma_{k}^{i}+\Gamma_{jk}^{r}\Gamma_{rh}^{i}-k|h]+\widetilde{C}_{jr}^{i}\widetilde{R}_{kh}^{r}$, $\widetilde{P}_{jkh}^{i}=\Gamma_{jk\|h}^{i}-\widetilde{C}_{jh|k}^{i}+\widetilde{C}_{jr}^{i}\widetilde{P}_{kh}^{r}$ and $\widetilde{S}_{jkh}^{i}=\widetilde{C}_{jk\|h}^{i}+\widetilde{C}_{jk}^{i}\widetilde{C}_{rk}^{i}-k|h$.

Note 4.2. $\widetilde{R}_{jkh}^{i}$, $\widetilde{P}_{jkh}^{i}$ and $\widetilde{S}_{jkh}^{i}$ are the h-curvature tensor, the hv-curvature tensor and the v-curvature tensor respectively on M_n.

Applying (4.11) to (4.9), we obtain

(4.12)

(a) $\widetilde{R}_{\alpha\sigma\beta\gamma}=g_{\sigma\varepsilon}\widetilde{R}_{\alpha\beta\gamma}^{\varepsilon}=\widetilde{R}_{jikh}B_{\alpha\sigma\beta\gamma}^{jikh}+B_{\alpha\sigma}^{ji}\{\widetilde{P}_{jikh}(B_{\beta}^{k}H_{\gamma}^{b}-B_{\gamma}^{k}H_{\beta}^{b})N_{b}^{h}$
$+\widetilde{S}_{jikh}N_{b}^{k}N_{c}^{h}H_{\beta}^{b}H_{\gamma}^{c}\}+[H_{\alpha\beta}^{b}(g_{jk|\gamma}B_{\sigma}^{j}N_{b}^{k}+\delta_{bc}H_{\sigma\gamma}^{c})-\beta|\gamma]$,

(b) $\widetilde{P}_{\alpha\sigma\beta\gamma}=g_{\sigma\varepsilon}\widetilde{P}_{\alpha\beta\gamma}^{\varepsilon}=\widetilde{P}_{jikh}B_{\alpha\sigma\beta\gamma}^{jikh}+\widetilde{S}_{jikh}B_{\alpha\sigma\gamma}^{jih}N_{b}^{k}H_{\beta}^{b}$
$+H_{\alpha\beta}^{b}(g_{ik}|_{\gamma}B_{\sigma}^{j}N_{b}^{k}+\delta_{bc}\widetilde{C}_{\sigma\beta}^{c})-\widetilde{C}_{\alpha\gamma}^{b}(g_{jk|\beta}B_{\sigma}^{j}N_{b}^{k}+\delta_{bc}H_{\sigma\beta}^{c})$,

(c) $\widetilde{S}_{\alpha\sigma\beta\gamma}=g_{\sigma\varepsilon}\widetilde{S}^{\varepsilon}_{\alpha\beta\gamma}=\widetilde{S}_{jikh}B^{jikh}_{\alpha\sigma\beta\gamma}+[\widetilde{C}^{\ \ b}_{\alpha\beta}(g_{jk}|_{\gamma}B^{j}_{\sigma}N^{k}_{b}+\delta_{bc}\widetilde{C}^{\ c}_{\sigma\gamma})-\beta|\gamma]$,

where $\widetilde{R}_{jikh}=g_{ir}\widetilde{R}^{\ r}_{j\ kh}$, $\widetilde{P}_{jikh}=g_{ir}\widetilde{P}^{\ r}_{j\ kh}$ and $\widetilde{S}_{jikh}=g_{ir}\widetilde{S}^{\ r}_{j\ kh}$.

§5. **Induced *TM*-connections.** The induced Matsumoto connection $IM\Gamma$ is called the *induced* TM(resp. $TM(0)$)*-connection* and denoted by $ITM\Gamma$(resp. $ITM\Gamma_o$), if the original connection $M\Gamma$ is a TM(resp. $TM(0)$)-connection. Then from (1. 7), (3. 8) and (3. 36) we have

$$\widetilde{C}^{\ \alpha}_{\beta\gamma}=C^{\ \alpha}_{\beta\gamma}\ (resp.\ \widetilde{C}^{\ \alpha}_{\beta\gamma}=0),\ \widetilde{C}^{\ \alpha}_{0\gamma}=C^{\ i}_{0k}B^{\alpha k}_{i\gamma}=0, \tag{5.1}$$

$$T^{0}_{\ \gamma}=T^{\alpha}_{\ 0}=0,\ Q^{\ 0}_{\beta\gamma}=Q^{\ \alpha}_{0\gamma}=0. \tag{5.2}$$

Because of (3. 40), (3. 42) and (5. 1) we obtain

$$\widetilde{T}^{\alpha}_{\ \gamma}=T^{\alpha}_{\ \gamma}-C^{\ \alpha}_{\gamma b}\overset{b}{H}{}^{b}_{\ 0},\ \widetilde{Q}^{\ \alpha}_{\beta\gamma}=Q^{\ \alpha}_{\beta\gamma}-C^{\ \alpha}_{\beta b}H^{b}_{\ \gamma}\ (resp.\ Q^{\ \alpha}_{\beta\gamma}-2C^{\ \alpha}_{\beta b}H^{b}_{\ \gamma}), \tag{5.3}$$

which, by virtue of (5. 2), implies

$$\widetilde{T}^{0}_{\ \gamma}=\widetilde{T}^{\alpha}_{\ 0}=0,\ \widetilde{Q}^{\ 0}_{\beta\gamma}=\widetilde{Q}^{\ \alpha}_{0\gamma}=0. \tag{5.4}$$

From Lemmas 3. 1, 3. 3, 3. 4, 3. 5 and (5. 1) it follows that the $ITM\Gamma$(resp. $ITM\Gamma_0$) satisfies axioms (F1), (F2), (F3)(resp. $(F3)_1$), (F4) and (F5). Therefore, taking accout of (5. 4), we can state

Theorem 5.1. *The $ITM\Gamma$ (*resp. *$ITM\Gamma_0$) is a TM(*resp. *$TM(0)$)-connection on M_m determined by the tensors $\widetilde{T}^{\alpha}_{\ \gamma}$, $\widetilde{Q}^{\ \alpha}_{\beta\gamma}$ and $\widetilde{C}^{\ \alpha}_{\beta\gamma}$ in* (5. 3) *and* (5. 1).

We put

$$T^{b}_{\ \gamma}=T^{i}_{\ k}N^{b}_{i}B^{k}_{\gamma}. \tag{5.5$_1$}$$

Then from (1. 7), (3. 13), (3. 38) and $(5.5)_1$ we have

$$H^{b}_{\ \gamma}=\overset{b}{H}{}^{b}_{\ \gamma}+T^{b}_{\ \gamma},\ H^{b}_{\ 0}=\overset{b}{H}{}^{b}_{\ 0}. \tag{5.5$_2$}$$

Therefore it follows from (5. 1) and $(5.5)_2$ that the equations in $(3.43)_3$ reduce to

$$C^{\ \alpha}_{\beta b}H^{b}_{\ \gamma}=0. \tag{5.6}$$

Consequently from Theorem 3. 2 we can state

Theorem 5.2. *The $ITM\Gamma$(or $ITM\Gamma_0$) is the intrinsic connection on M_m if and only if an equation* (5. 6) *holds.*

From (3. 9), (3. 11) and (5. 1) we obtain

(5. 7)$_1$ $$\overset{\circ}{\Gamma}{}^{a}_{\beta\gamma} := B^{a}_{i}(B^{i}_{\beta\gamma} + \Gamma^{i}_{jk} B^{jk}_{\beta\gamma}),\quad \Gamma^{a}_{\beta\gamma} = \overset{\circ}{\Gamma}{}^{a}_{\beta\gamma} + C^{a}_{\beta b} H^{b}_{\gamma},$$

(5. 7)$_2$ $$y^{\beta} \overset{\circ}{\Gamma}{}^{a}_{\beta\gamma} = y^{\beta} \Gamma^{a}_{\beta\gamma} = \Gamma^{a}_{\gamma} = B^{a}_{i}(B^{i}_{0\gamma} + \Gamma^{i}_{k} B^{k}_{\gamma}).$$

If we denote the h-torsion tensors of $ITM\Gamma_0$ and ITMΓ by $\overset{\circ}{\tau}{}^{a}_{\beta\gamma}$ and $\widetilde{\tau}^{a}_{\beta\gamma}$ respectively, then from (5. 7)$_1$ we have

(5. 8) $$\overset{\circ}{\tau}{}^{a}_{\beta\gamma} = \tau^{a}_{\beta\gamma} = \tau^{i}_{jk} B^{ajk}_{i\beta\gamma},\quad \widetilde{\tau}^{a}_{\beta\gamma} = \overset{\circ}{\tau}{}^{a}_{\beta\gamma} + (C^{a}_{\beta b} H^{b}_{\gamma} - C^{a}_{\gamma b} H^{b}_{\beta}).$$

Differentiating $\widetilde{T}^{a}{}_{\gamma}$ in (5. 3) by y^{β}, from (2. 20), (2. 21), (2. 27) and Lemma 2. 2 we obtain

(5. 9) $$\widetilde{T}^{a}_{\beta\gamma} = \widetilde{T}^{a}{}_{\gamma\|\beta} = T^{a}_{\beta\gamma} + 2C^{a}_{\beta b} T^{b}{}_{\gamma} - (2C^{a}_{\gamma b} \overset{b}{H}{}^{b}_{\beta} + \mu_{b}{}^{a}{}_{\beta\gamma} \overset{b}{H}{}^{b}_{0}),$$

where $T^{a}_{\beta\gamma} = T^{i}_{jk} B^{ajk}_{i\beta\gamma}$. Similarly from (3. 37) and (3. 39) we have

(5. 10) $$G^{a}_{\beta\gamma} = G^{a}{}_{\gamma\|\beta} = \overset{b}{\Gamma}{}^{a}_{\beta\gamma} + 2C^{a}_{\beta b} \overset{b}{H}{}^{b}_{\gamma} + (2C^{a}_{\gamma b} \overset{b}{H}{}^{b}_{\beta} + \mu_{b}{}^{a}{}_{\beta\gamma} \overset{b}{H}{}^{b}_{0}),$$

where $G^{a}_{\beta\gamma}$ is the intrinsic h-connection of Berwald.

In this case, it is easily verified from (5. 3), (5. 9) and (5. 10) that

$$G^{a}_{\beta\gamma} + \widetilde{T}^{a}_{\beta\gamma} + \widetilde{Q}^{a}_{\beta\gamma} = \Gamma^{a}_{\beta\gamma} (\text{resp. } \overset{\circ}{\Gamma}{}^{a}_{\beta\gamma}).$$

We shall denote the induced connections on M_m from $B\Gamma$, $H\Gamma$, RT and $C\Gamma$ by $IB\Gamma$, $IH\Gamma$, $IR\Gamma$ and $IC\Gamma$. Then from (1. 8), (3. 36) and (5. 3) we have

(5. 11) $$\widetilde{T}^{a}{}_{\gamma} = -C^{a}_{\gamma b} \overset{b}{H}{}^{b}_{0} \qquad \text{for any of } IB\Gamma,\ IH\Gamma,\ IR\Gamma \text{ and } IC\Gamma,$$

(5. 12)

(a) $\widetilde{Q}^{a}_{\beta\gamma} = -2C^{a}_{\beta b} \overset{b}{H}{}^{b}_{\gamma},\ \widetilde{C}^{a}_{\beta\gamma} = 0$ for $IB\Gamma$,

(d) $\widetilde{Q}^{a}_{\beta\gamma} = -C^{a}_{\beta b} \overset{b}{H}{}^{b}_{\gamma},\ \widetilde{C}^{a}_{\beta\gamma} = C^{a}_{\beta\gamma}$ for $IH\Gamma$,

(c) $\widetilde{Q}^{a}_{\beta\gamma} = -P^{a}_{\beta\gamma} - 2C^{a}_{\beta b} \overset{b}{H}{}^{b}_{\gamma},\ \widetilde{C}^{a}_{\beta\gamma} = 0$ for $IR\Gamma$,

(d) $\widetilde{Q}^{a}_{\beta\gamma} = -P^{a}_{\beta\gamma} - C^{a}_{\beta b} \overset{b}{H}{}^{b}_{\gamma},\ \widetilde{C}^{a}_{\beta\gamma} = C^{a}_{\beta\gamma}$ for $IC\Gamma$,

where $P^{a}_{\beta\gamma} = P^{i}_{jk} B^{ajk}_{i\beta\gamma}$.

Differentiating (5. 11) by y^{γ}, we have

(5. 13) $$\widetilde{T}^{a}_{\beta\gamma} = -2C^{a}_{\gamma b} \overset{b}{H}{}^{b}_{\beta} - \mu_{b}{}^{a}{}_{\beta\gamma} \overset{b}{H}{}^{b}_{0}.$$

From (5. 10) we get

(5.14) $\overset{b}{\Gamma}{}^{\alpha}_{\beta\gamma} = G^{\alpha}_{\beta\gamma} - 2(C^{\alpha}_{\beta b}\overset{b}{H}{}^{b}_{\gamma} + C^{\alpha}_{\gamma b}\overset{b}{H}{}^{b}_{\beta}) - \mu^{\alpha}_{b\beta\gamma}\overset{b}{H}{}^{b}_{0}.$

We shall denote the h-connections of $IH\Gamma$, $IR\Gamma$ and $IC\Gamma$ by $\overset{h}{\Gamma}{}^{\alpha}_{\beta\gamma}$, $\overset{r}{\Gamma}{}^{\alpha}_{\beta\gamma}$ and $\overset{c}{\Gamma}{}^{\alpha}_{\beta\gamma}$ respectively. Then it follows from (5.12) and (5.13) that

(5.15) $\overset{h}{\Gamma}{}^{\alpha}_{\beta\gamma} = G^{\alpha}_{\beta\gamma} - (C^{\alpha}_{\beta b}\overset{b}{H}{}^{b}_{\gamma} + 2C^{\alpha}_{\gamma b}\overset{b}{H}{}^{b}_{\beta} + \mu^{\alpha}_{b\beta\gamma}\overset{b}{H}{}^{b}_{0}),$

(5.16) $\overset{r}{\Gamma}{}^{\alpha}_{\beta\gamma} = G^{\alpha}_{\beta\gamma} - (P^{\alpha}_{\beta\gamma} + 2C^{\alpha}_{\beta b}\overset{b}{H}{}^{b}_{\gamma} + 2C^{\alpha}_{\gamma b}\overset{}{H}{}^{b}_{\beta} + \mu^{\alpha}_{b\beta\gamma}\overset{b}{H}{}^{b}_{0}),$

(5.17) $\overset{c}{\Gamma}{}^{\alpha}_{\beta\gamma} = G^{\alpha}_{\beta\gamma} - (P^{\alpha}_{\beta\gamma} + C^{\alpha}_{\beta b}\overset{b}{H}{}^{b}_{\gamma} + 2C^{\alpha}_{\gamma b}\overset{b}{H}{}^{b}_{\beta} + \mu^{\alpha}_{b\beta\gamma}\overset{b}{H}{}^{b}_{0}).$

Thus taking account of Theorem 5.1 and Theorem 5.2, we can state

Corollary 5.1.1. *The induced typical connections $IB\Gamma$, $IH\Gamma$, $IR\Gamma$ and $IC\Gamma$ are TM(or $TM(0)$)-connections on M_m determined by tensors $\widetilde{T}^{\alpha}{}_{\gamma}$, $\widetilde{Q}^{\alpha}_{\beta\gamma}$ and $\widetilde{C}^{\alpha}_{\beta\gamma}$ in* (5.11) *and* (5.12), *and their h-connections are expressible in* (5.14) ~(5.17), *while the their non-linear connections are commonly equal to $\overset{b}{\Gamma}{}^{\alpha}{}_{\gamma}$. The above four induced connections are the intrinsic connections on M_m respectively if and only if the same equation $C^{\alpha}_{\beta b}\overset{b}{H}{}^{b}_{\gamma} = 0$ holds.*

We shall call the $ITM\Gamma$ the *induced RTM-connection* if the original $TM\Gamma$ is an RTM-connection, and denote it by $IRTM\Gamma$.

Since an $RTM\Gamma$ is an h-metrical TM-connection, it follows from Lemma 3.2 and Theorem 5.1 that the $IRTM\Gamma$ satisfies axioms (F2), (F3), (F5) and (F6), that is, it is an $RTM\Gamma$ on M_m. If we put

$$\widetilde{Q}_{\alpha\beta\gamma} = g_{\beta\sigma}\widetilde{Q}^{\sigma}_{\alpha\gamma},\quad \widetilde{Z}_{\alpha\beta\gamma} = \widetilde{Q}_{\alpha\beta\gamma} - \widetilde{Q}_{\beta\alpha\gamma},$$

then from (5.3) we have

(5.18) $\widetilde{Z}_{\alpha\beta\gamma} = Z_{ijk}B^{ijk}_{\alpha\beta\gamma},\quad Z_{ijk} = Q_{ijk} - Q_{jik},$

where $Q_{ijk} = g_{jh}Q^{h}_{ik}$. Since $Q_{0jk} = Q_{i0k} = 0$, it follows from (5.18) that $\widetilde{Z}_{\alpha\beta\gamma} + \widetilde{Z}_{\beta\alpha\gamma} = 0$ and $\widetilde{Z}_{0\beta\gamma} = \widetilde{Z}_{\alpha 0\gamma} = 0$. Therefore, taking account of (1.10), we obtain

(5.19) $\Gamma^{\alpha}_{\beta\gamma} = \Gamma^{*\alpha}_{\beta\gamma} - C^{\alpha}_{\beta\sigma}\widetilde{T}^{\sigma}{}_{\gamma} + \frac{1}{2}g^{\alpha\sigma}(\widetilde{Z}_{\beta\sigma\gamma} + \widetilde{T}_{\beta\sigma\gamma} - \widetilde{T}_{\sigma\beta\gamma}),$

where $\Gamma^{*\alpha}_{\beta\gamma}$ is the intrinsic h-connection of Cartan and $\widetilde{T}_{\beta\sigma\gamma} = g_{\sigma\alpha}\widetilde{T}^{\alpha}_{\beta\gamma}$.

Thus we can state

Theorem 5.3. *The induced connection $IRTM\Gamma$ is an RTM-connection on M_m determind by tensors $\widetilde{T}^{\alpha}{}_{\gamma}$, $\widetilde{Z}_{\alpha\beta\gamma}$(in* (5.3), (5.18)) *and $C^{\alpha}_{\beta\gamma}$, and the h-connection is ex-*

pressible in (5. 19).

In case of *ICΓ*, it follows from (1) in (1. 11), (2. 29), (5. 9) and (5. 18) that

$$\widetilde{T}_{\beta\sigma\gamma} = -2C_{\gamma\sigma a}\overset{b}{H}{}^{a}_{\beta} - \{v_{a\sigma\beta\gamma} - 2(C_{\gamma\sigma\varepsilon}C^{\varepsilon}_{a\beta} + C_{\beta\sigma\varepsilon}C^{\varepsilon}_{a\gamma})\}\overset{b}{H}{}^{a}_{0}, \tag{5.20}$$

$$\widetilde{T}^{\sigma}{}_{\gamma} = -C^{\sigma}_{\gamma a}\overset{b}{H}{}^{a}_{0},\ \widetilde{Z}_{\alpha\beta\gamma} = 0. \tag{5.21}$$

Applying (5. 20) and (5. 21) to (5. 19) and using Lemma 2. 2, we obtain

$$\begin{aligned}\overset{c}{\Gamma}{}^{a}_{\beta\gamma} &= \Gamma{*}^{a}_{\beta\gamma} + U^{a}_{\beta\gamma} + C_{\beta\gamma b}\overset{b}{H}{}^{ba} - C^{a}_{\gamma b}\overset{b}{H}{}^{b}_{\beta},\ \overset{b}{H}{}^{ba} = g^{\sigma a}\overset{b}{H}{}^{b}_{\sigma},\\ U^{a}_{\beta\gamma} &= (C^{a}_{\beta\sigma}C^{\sigma}_{b\gamma} + C^{a}_{\gamma\sigma}C^{\sigma}_{b\beta} - C^{\sigma}_{\beta\gamma}C^{a}_{b\sigma})\overset{b}{H}{}^{b}_{0},\end{aligned} \tag{5.22}$$

If we denote the *hv*-torsion tensor with respect to the intrinsic Cartan connection by $P{*}^{a}_{\beta\gamma}$, then from (5. 17) and (5. 22) we have

$$\begin{aligned}P{*}^{a}_{\beta\gamma} &= G^{a}_{\beta\gamma} - \Gamma{*}^{a}_{\beta\gamma}\\ &= P^{a}_{\beta\gamma} + C^{a}_{\beta b}\overset{b}{H}{}^{b}_{\gamma} + C^{a}_{\gamma b}\overset{b}{H}{}^{b}_{\beta} + C_{\beta\gamma b}\overset{b}{H}{}^{ba} + U^{a}_{\beta\gamma} + \mu^{a}_{b\beta\gamma}\overset{b}{H}{}^{b}_{0}.\end{aligned} \tag{5.23}$$

Applying (5. 23) to (5. 16), we obtain

$$\overset{r}{\Gamma}{}^{a}_{\beta\gamma} = \Gamma{*}^{a}_{\beta\gamma} + U^{a}_{\beta\gamma} + C_{\beta\gamma b}\overset{b}{H}{}^{ba} - C^{a}_{\beta b}\overset{b}{H}{}^{b}_{\gamma} - C^{a}_{\gamma b}\overset{b}{H}{}^{b}_{\beta}. \tag{5.24}$$

Thus we can state

Corollary 5.3.1. *The ICΓ is an RTM-connection on* M_m *determined by tensors* $\widetilde{T}^{a}{}_{\gamma}$, $\widetilde{Z}_{\alpha\beta\gamma}$ *(in* (5. 21)) *and* $C^{a}_{\beta\gamma}$, *and the h-connection is also expressible in* (5. 22). *The hv-torsion tensor* $P{*}^{a}_{\beta\gamma}$ *with respect to the intrinsic Cartan connection is given by* (5. 23). *The h-connection of IRΓ is also expressible in* (5. 24).

Note 5.1. The *h*-connections (5. 14) and (5. 15) are also expressible in terms of $\Gamma{*}^{a}_{\beta\gamma}$ by means of (5. 23).

From (5. 22) and Lemma 3. 6 we can state

Corollary 5.3.2. For *ICΓ*, $\Gamma^{a}{}_{\gamma} = G^{a}{}_{\gamma}$ and $\overset{c}{\Gamma}{}^{a}_{\beta\gamma} = \Gamma{*}^{a}_{\beta\gamma}$ if and only if any one of facts (1) ~(4) in Lemma 3. 6 holds.

Note 5.2. Since the *hv*-torsion with respect to *CΓ* is given by $P^{i}_{j\,k}$, the *hv*-torsion tensor with respect to the intrinsic Cartan connection should be given by $P{*}^{a}_{\beta\gamma} = P^{a}_{\beta\gamma}$. In this case, we have $C^{a}_{\beta b}\overset{b}{H}{}^{b}_{\gamma} = 0$ from (5. 23).

For the induced *IS*-connection,[8] it follows from (b) in (1. 11), (5. 3), (5. 9) and

8) The *IMΓ* is called the *induced IS-connection* if the *MΓ* is the *IS*-connection.

(5. 18) that

$$(5.25)\qquad \widetilde{T}^{a}{}_{\gamma}=T^{a}{}_{\gamma}-C^{\ a}_{\gamma b}\overset{b}{H}{}^{b}_{o},\ \widetilde{Z}_{\alpha\beta\gamma}=0,\ \widetilde{C}^{\ a}_{\beta\gamma}=C^{\ a}_{\beta\gamma},$$

$$(5.26)\qquad \begin{aligned}&T_{\alpha\beta\gamma}+T_{\beta\delta\gamma}+2(C_{\alpha\beta\sigma}T^{\sigma}{}_{\gamma}+C_{\alpha\beta b}T^{b}{}_{\gamma}+P_{\alpha\beta\gamma})=0,\\ &\widetilde{T}_{\beta\sigma\gamma}=T_{\beta\sigma\gamma}+2C_{\beta\sigma b}T^{b}{}_{\gamma}-2C_{\gamma\sigma b}\overset{b}{H}{}^{b}_{\beta}-\mu_{b\sigma\beta\gamma}\overset{b}{H}{}^{b}_{o}.\end{aligned}$$

If we denote the *h*-connection of the above by $\overset{t}{\Gamma}{}^{a}_{\beta\gamma}$, then from (2. 29), (5. 19), (2. 25) and (5. 26) we obtain

$$(5.27)\qquad \overset{t}{\Gamma}{}^{a}_{\beta\gamma}=\Gamma^{*a}_{\beta\gamma}+T^{a}_{\beta\gamma}+P^{a}_{\beta\gamma}+C^{\ a}_{\beta b}T^{b}{}_{\gamma}+C_{\beta\gamma b}\overset{b}{H}{}^{ba}-C^{\ a}_{\gamma b}\overset{b}{H}{}^{b}_{\beta}+U^{\ a}_{\beta\gamma}.$$

Consequently we can state

Corollary 5.3.3. *The induced IS-connection is an RTM-connection on M_m determined by tensors $\widetilde{T}^{a}{}_{\gamma}$, $\widetilde{Z}_{\alpha\beta\gamma}$ and $\widetilde{C}^{\ a}_{\beta\gamma}$ in* (5. 25) *together with* (5. 26), *and the h-connection is expressible in* (5. 27).

Note 5.3. In view of (5. 22), (5. 27) is changeable to

$$(5.28)\qquad \overset{t}{\Gamma}{}^{a}_{\beta\gamma}=\overset{c}{\Gamma}{}^{a}_{\beta\gamma}+T^{a}_{\beta\gamma}+P^{a}_{\beta\gamma}+C^{\ a}_{\beta b}T^{b}_{\gamma}.$$

For the induced *AMR*-connection,[9] from (c) in (1. 11), (5. 3), (5. 9) and (5. 18) we obtain

$$(5.29)\qquad \begin{aligned}&\widetilde{T}^{a}{}_{\gamma}=\bar{f}\bar{L}h^{a}{}_{\gamma}-C^{\ a}_{\gamma b}\overset{b}{H}{}^{b}_{o},\ \widetilde{C}^{\ a}_{\beta\gamma}=C^{\ a}_{\beta\gamma},\\ &\widetilde{Z}_{\alpha\beta\gamma}=\bar{L}(\bar{f}_{\mid\beta}h_{\alpha\gamma}-\bar{f}_{\mid\alpha}h_{\beta\gamma}),\end{aligned}$$

where $\bar{f}$ is the scalar induced on M_m from f on M_n and $h^{a}{}_{\gamma}=\delta^{a}{}_{\gamma}-l^{a}l_{\gamma}$,

$$(5.30)\qquad \begin{aligned}\widetilde{T}_{\beta\sigma\gamma}=\bar{f}(l_{\beta}h_{\sigma\gamma}-l_{\gamma}h_{\sigma\beta}-l_{\sigma}h_{\beta\gamma})+\bar{L}\bar{f}_{\mid\beta}h_{\sigma\gamma}\\ +2C_{\gamma\sigma b}\overset{b}{H}{}^{b}_{\beta}-\mu_{b\sigma\beta\gamma}\overset{b}{H}{}^{b}_{o}.\end{aligned}$$

Applying (5. 29) and (5. 30) to (5. 19) and using (5. 22) we have

$$(5.31)\qquad \Gamma^{a}_{\beta\gamma}=\overset{c}{\Gamma}{}^{a}_{\beta\gamma}+\bar{f}(l_{\beta}h^{a}{}_{\gamma}-l^{a}h_{\beta\gamma}-\bar{L}C^{\ a}_{\beta\gamma}).$$

Hence we can state

Corollary 5.3.4. *The induced AMR-connection is an RTM-connection on M_m determined by tensors $\widetilde{T}^{a}{}_{\gamma}$, $\widetilde{Z}_{\alpha\beta\gamma}$ and $\widetilde{C}^{\ a}_{\beta\gamma}$ in* (5. 29), *and the h-connection is expressible*

9) The *IMΓ* is called the *induced AMR-connection* if the *MΓ* is an *AMRΓ*.

in (5. 31).

We shall call the *ITMΓ* (resp. *ITMΓ*$_o$) the *induced STM*(resp. *STM*(0))*-connection* and denote it by *ISTMΓ*(resp. *ISTMΓ*$_o$), if the original *TMΓ*(resp. *TM* Γ_o) is *h*-symmetric. In this case, from (5. 8) we have

$$(5.32)\qquad \overset{o}{\tau}{}^a_{\beta\gamma}=0,\ \tilde{\tau}^a_{\beta\gamma}=C^a_{\beta b}H^b_\gamma-C^a_{\gamma b}H^b_\beta.$$

which implies that the tensor $\tilde{\tau}^a_{\beta\gamma}$ vanishes if and only if the following holds :

$$(5.33)\qquad C^a_{\beta b}H^b_\gamma=C^a_{\gamma b}H^b_\beta.$$

Contracting (5. 33) by y^γ, we obtain $C^a_{\beta b}\overset{b}{H}{}^b_o=0$. Therefore it follows from (5. 33) and (4) in Lemma 3. 6 that the equation (5. 33) holds if and only if the following equations hold :

$$(5.34)\qquad C^a_{\beta b}\overset{b}{H}{}^b_o=0,\ C^a_{\beta b}T^b_\gamma=C^a_{\gamma b}T^b_\beta.$$

Consequently we can state

Lemma 5.1. *The connection ISTMΓis an STMΓ on* M_m *if and only if* (5. 34) *holds.*

The *h*-connection in (1. 15) is expressible in

$$(5.35)\qquad \Gamma^i_{jk}=G^i_{jk}+T^i_{jk}(=\tfrac{1}{2}Q^i_{kOIj})+\overline{Q}{}^i_{jk},$$

where $\overline{Q}{}^i_{jk}=\tfrac{1}{2}(Q^i_{jk}+Q^i_{kj})+\tfrac{1}{4}(Q^i_{jOIk}-Q^i_{kOIj})$. Then we can state

Lemma 5.2. *The tensor* $\overline{Q}{}^i_{jk}$ *coincides with* Q^i_{jk} *if and only if*

$$(5.36)\qquad Q^i_{jk}-Q^i_{kj}=\tfrac{1}{2}(Q^i_{jOIk}-Q^i_{kOIj}).$$

We shall call an *STMΓ*(resp. *STMΓ*$_o$) an S_q*TM*(resp. S_q*TM*(0))*-connection* and denote it by S_q*TMΓ*(resp. S_q*TMΓ*$_o$), if it is determined by a tensor Q^i_{jk} satisfying (5. 36). In this case, the induced connection on M_m from the above will be denoted by *IS*$_q$*TMΓ*(resp. *IS*$_q$*TMΓ*$_o$).

Note 5.4. The connections *CΓ*, *HΓ*, *AMBΓ* and *AMCΓ* are S_q*TM*-connections, while the connections *RΓ*, *BΓ*, *AMBΓ*$_o$ and *AMCΓ*$_o$ are S_q*TM*(0)-connections.

If we put $Q^b_{\gamma o}=Q^i_{ko}B^k_\gamma N^b_i$, then from (2. 20) we have

$$(5.37)\qquad Q^i_{kOIj}B^{ajk}_{i\beta\gamma}=Q^a_{\gamma OI\beta}=2C^a_{\beta b}Q^b_{\gamma o}.$$

Let the *IS*$_q$*TMΓ* be *h*-symmetric. Then it follows from (1. 14), (5. 5)$_2$, (5. 14),

(5. 35), (5. 37), Lemma 3. 6 and Lemma 5.1 that the h-connection is expressible in

$$\text{(5. 38)}\quad \begin{aligned}\Gamma^{a}_{\beta\gamma} &= \overset{b}{\Gamma}{}^{a}_{\beta\gamma} + (\tfrac{1}{2}Q^{i}_{k0|j} + Q^{i}_{jk})B^{ajk}_{i\beta\gamma} + C^{a}_{\beta b}H^{b}_{\gamma}\\ &= G^{a}_{\beta\gamma} + \tfrac{1}{2}\widetilde{Q}^{a}_{\gamma 0|\beta} + \widetilde{Q}^{a}_{\beta\gamma},\end{aligned}$$

where $\widetilde{Q}^{a}_{\beta\gamma} = Q^{a}_{\beta\gamma} - C^{a}_{\beta b}(\overset{b}{H}{}^{b}_{\gamma} + \tfrac{1}{2}Q^{b}_{\gamma 0}) = Q^{a}_{\beta\gamma} - C^{a}_{\beta b}H^{b}_{\gamma}$ and $\widetilde{Q}^{a}_{\gamma 0} = Q^{a}_{\gamma 0}$.

From (5. 34), (5. 36) and (5. 37) we obtain

$$\text{(5. 39)}\quad Q^{a}_{\beta\gamma} - Q^{a}_{\gamma\beta} = \tfrac{1}{2}(Q^{a}_{\beta 0|\gamma} - Q^{a}_{\gamma 0|\beta}).$$

Then, from (3. 30) and (5. 39) we get

$$\text{(5. 40)}\quad \widetilde{Q}^{a}_{\beta\gamma} - \widetilde{Q}^{a}_{\gamma\beta} = \tfrac{1}{2}(Q^{a}_{\beta 0|\gamma} - Q^{a}_{\beta 0|\beta}) = \tfrac{1}{2}(\widetilde{Q}^{a}_{\beta 0|\gamma} - \widetilde{Q}^{a}_{\gamma 0|\beta}).$$

Consequently, in view of (5. 40) we can state

Theorem 5.4. *If the induced connection $IS_qTM\Gamma$ is h-symmetric, then it is an $S_qTM\Gamma$ on M_m determined by a tensor $\widetilde{Q}^{a}_{\beta\gamma}$ in* (5. 38).

For the induced $AMB\Gamma$(or $AMC\Gamma$)[10], from (1. 18), (2. 11) and $(5. 5)_1$ we have $T^{a}_{\gamma} = 0$. Therefore from (1. 16), (1. 17), (5. 38), Lemma 3. 6 and Lemma 5. 1 it follows that

$$\text{(5. 41)}\quad \widetilde{Q}^{a}_{\beta\gamma} = 2\bar{f}l_{\gamma}h^{a}_{\beta} - \bar{L}\bar{f}_{|\beta}h^{a}_{\gamma} - C^{a}_{\beta b}\overset{b}{H}{}^{b}_{\gamma},$$

for $IAMB\Gamma$

$$\text{(5. 42)}\quad \Gamma^{a}_{\beta\gamma} = G^{a}_{\beta\gamma} + \bar{f}(l_{\beta}h^{a}_{\gamma} + l_{\gamma}h^{a}_{\beta} - l^{a}h_{\beta\gamma}) - C^{a}_{\beta b}\overset{b}{H}{}^{b}_{\gamma},$$

$$\text{(5. 43)}\quad \widetilde{Q}^{a}_{\beta\gamma} = 2\bar{f}l_{\gamma}h^{a}_{\beta} - \bar{L}\bar{f}_{|\beta}h^{a}_{\gamma} - C^{a}_{\beta b}\overset{b}{H}{}^{b}_{\gamma} - P^{a}_{\beta\gamma},$$

for $IAMC\Gamma$

$$\text{(5. 44)}\quad \Gamma^{a}_{\beta\gamma} = \Gamma^{*a}_{\beta\gamma} + \bar{f}(l_{\beta}h^{a}_{\gamma} + l_{\gamma}h^{a}_{\beta} - l^{a}h_{\beta\gamma}).$$

Hence we can state

Corollary 5.4.1. *If the induced AMB*(resp. *AMC*)*-connection is h-symmetric, then it is an $S_qTM\Gamma$ on M_m determined by a tensor $\widetilde{Q}^{a}_{\beta\gamma}$ in* (5. 41)(*resp.* (5. 43)), *and the h-connection is given by* (5. 42) (*resp.* (5. 44)).

From the first in (5. 32) it is seen that the $ISTM\Gamma_o$ is h-symmetric. In particular, for the $IS_qTM\Gamma_o$ we put

$$\text{(5. 45)}\quad \widetilde{Q}^{a}_{\beta\gamma} = Q^{a}_{\beta\gamma} - 2C^{a}_{\beta b}\overset{b}{H}{}^{b}_{\gamma} - C^{a}_{\beta b}Q^{b}_{\gamma 0}.$$

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10) The $IM\Gamma$ is called the *induced AMB*(resp. *AMC*)*-connection* if the $M\Gamma$ is an $AMB\Gamma$(resp. $AMC\Gamma$).

Then the h-connection is expressible in

$$(5.46)\qquad \begin{aligned}\overset{0}{\Gamma}{}^{\alpha}_{\beta\gamma} &= \overset{b}{\Gamma}{}^{\alpha}_{\beta\gamma} + \tfrac{1}{2} Q^{i}_{k0|j} B^{\alpha jk}_{i\beta\gamma} + Q^{\alpha}_{\beta\gamma} \\ &= G^{\alpha}_{\beta\gamma} + \tfrac{1}{2}\widetilde{Q}^{\alpha}_{\gamma0|\beta} + \widetilde{Q}^{\alpha}_{\beta\gamma}.\end{aligned}$$

Further it is proved that the following relation holds :

$$(5.48)\qquad \widetilde{Q}^{\alpha}_{\beta\gamma} - \widetilde{Q}^{\alpha}_{\gamma\beta} = \tfrac{1}{2}(\widetilde{Q}^{\alpha}_{\beta0|\gamma} - \widetilde{Q}^{\alpha}_{\gamma0|\beta}).$$

Consequently we can state

Theorem 5.5. *The induced connection $IS_qTM\Gamma_0$ is an $S_qTM\Gamma_0$ on M_m determined by a tensor $\widetilde{Q}^{\alpha}_{\beta\gamma}$ in* (5. 45).

For the induced $AMB\Gamma_0$ (or $AMC\Gamma_0$)[11], we have also $T^{\alpha}_{\gamma}=0$. Therefore, as before we obtain

$$(5.49)\qquad \widetilde{Q}^{\alpha}_{\beta\gamma} = 2\bar{f} l_{\gamma} h^{\alpha}_{\beta} - L\bar{f}_{|\beta} h^{\alpha}_{\gamma} - 2C^{\alpha}_{\beta b}\overset{b}{H}{}^{b}_{\gamma},$$

for $IAMB\Gamma_0$

$$(5.50)\qquad \Gamma^{\alpha}_{\beta\gamma} = \overset{b}{\Gamma}{}^{\alpha}_{\beta\gamma} + \bar{f}(l_{\beta} h^{\alpha}_{\gamma} + l_{\gamma} h^{\alpha}_{\beta} - l^{\alpha} h_{\beta\gamma}),$$

$$(5.51)\qquad \widetilde{Q}^{\alpha}_{\beta\gamma} = 2\bar{f} l_{\gamma} h^{\alpha}_{\beta} - L\bar{f}_{|\beta} h^{\alpha}_{\gamma} - P^{\alpha}_{\beta\gamma} - 2C^{\alpha}_{\beta b}\overset{b}{H}{}^{b}_{\gamma},$$

for $IAMC\Gamma_0$

$$(5.52)\qquad \Gamma^{\alpha}_{\beta\gamma} = \overset{r}{\Gamma}{}^{\alpha}_{\beta\gamma} + \bar{f}(l_{\beta} h^{\alpha}_{\gamma} + l_{\gamma} h^{\alpha}_{\beta} - l^{\alpha} h_{\beta\gamma}).$$

Hence we can state

Corollary 5.5.1. *The induced AMB* (0) (*resp. AMC* (0))*-connection is an $S_qTM\Gamma_0$ on M_m determined by a tensor $\widetilde{Q}^{\alpha}_{\beta\gamma}$ in* (5. 49) (*resp.* (5. 51)), *and the h-connection is given by* (5. 50) (*resp.* (5. 52)).

§ 6. Induced *TMD*-connections. The induced connection $IM\Gamma$ is called the *induced TMD*(resp. *TMD*(0))*-connection* and denoted by $ITMD\Gamma$(resp. $ITMD\Gamma_0$), if the original connection $M\Gamma$ is a *TMD*(resp. *TMD*(0))-connection. Then from (1. 21) and (3. 36) we have

$$(6.1)\qquad \widetilde{C}^{\alpha}_{\beta\gamma} = C^{\alpha}_{\beta\gamma}\ (\text{resp. } \widetilde{C}^{\alpha}_{\beta\gamma}=0),\quad D^{\alpha}_{\gamma} = D^{i}_{k} B^{\alpha k}_{i\gamma} = Q^{i}_{0k} B^{\alpha k}_{i\gamma},$$

$$(6.2)\qquad T^{0}_{\gamma} = T^{\alpha}_{0} = 0,\quad D_{\beta\gamma} + Q_{\beta0\gamma} = 0,$$

where $Q_{\beta0\gamma} = Q^{0}_{\beta\gamma}$ and $D_{\beta\gamma} = g_{\beta\alpha} D^{\alpha}_{\gamma}$.

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11) The $IM\Gamma$ is called the *induced AMB*(0) (resp. *AMC*(0))*-connection* if the $M\Gamma$ is an $AMB\Gamma_0$(resp. $AMC\Gamma_0$).

As before we obtain

$$(6.3)\qquad \widetilde{T}^{a}{}_{\gamma}=T^{a}{}_{\gamma}-C^{\ a}_{\gamma b}\overset{b}{H}{}^{b}_{\ o},\quad \widetilde{Q}^{\ a}_{\beta\gamma}=Q^{\ a}_{\beta\gamma}-C^{\ a}_{\beta b}H^{\ b}_{\gamma}\quad (resp.\ Q^{\ a}_{\beta\gamma}-2C^{\ a}_{\beta b}H^{\ b}_{\gamma}).$$

Applying (6. 1) to (3. 28), we have $\widetilde{D}^{a}{}_{\gamma}=D^{a}{}_{\gamma}$. Therefore it follows from (6. 2) and (6. 3) that

$$(6.4)\qquad \widetilde{T}^{\ o}_{\gamma}=\widetilde{T}^{a}{}_{o}=0,\ \widetilde{D}_{\beta\gamma}+\widetilde{Q}_{\beta o\gamma}=0.$$

Consequently by virtue of (6. 4) we can state

Theorem 6.1. *The ITMDΓ(resp. ITMDΓ_o) is a TMDΓ(resp. TMDΓ_o) on M_m determined by tensors $\widetilde{T}^{a}{}_{\gamma}$, $\widetilde{Q}^{\ a}_{\beta\gamma}$ and $\widetilde{C}^{\ a}_{\beta\gamma}$ in* (6. 3) *and* (6. 1).

In the above case, the relations $(5.5)_1$ and $(5.5)_2$ are still valid. Therefore we can state

Theorem 6.2. *The ITMDΓ(or ITMDΓ_o) is the intrinsic connection on M_m if and only if an equation $C^{\ a}_{\beta b}H^{\ b}_{\gamma}=0$ holds.*

The connection *IMΓ* is called the *induced STD*(resp. *STD*(0))*-connection* and denoted by *ISTDΓ*(resp. *ISTD*Γ_o), if the original connection *MΓ* is an *STDΓ*(resp. *STD*Γ_o). Then it follows from (1. 22) that the *h*-connection $\Gamma^{\ a}_{\beta\gamma}$(resp.$\overset{o}{\Gamma}{}^{\ a}_{\beta\gamma}$) is given by

$$(6.5)\qquad \begin{aligned}&\Gamma^{\ a}_{\beta\gamma}=\overset{c}{\Gamma}{}^{\ a}_{\beta\gamma}+C^{\ a}_{\beta b}T^{b}{}_{\gamma}+W^{\ a}_{\beta\gamma},\ \overset{o}{\Gamma}{}^{\ a}_{\beta\gamma}=\overset{r}{\Gamma}{}^{\ a}_{\beta\gamma}+W^{\ a}_{\beta\gamma},\\ &W^{\ a}_{\beta\gamma}=(C_{jkr}T^{ri}-C^{\ i}_{jr}T^{r}{}_{k}-C^{\ i}_{kr}T^{r}{}_{j})B^{ajk}_{i\beta\gamma}.\end{aligned}$$

In this case, the non-linear connection is commonly given by $\Gamma^{a}{}_{\gamma}=G^{a}{}_{\gamma}+\widetilde{T}^{a}{}_{\gamma}$. The tensor $\widetilde{T}^{a}{}_{\gamma}$ satisfies $\widetilde{T}^{\ o}_{\gamma}=\widetilde{T}^{a}{}_{o}=0$ because of (6. 4). Therefore by means of this tensor, an *STDΓ*(or *STD*Γ_o) on M_m is uniquely determined. If we denote the *h*-connection of this connection by $\widetilde{\Gamma}^{\ a}_{\beta\gamma}$, then according to (1. 22) we have

$$(6.6)\qquad \widetilde{\Gamma}^{\ a}_{\beta\gamma}=\Gamma^{*\ a}_{\ \beta\gamma}+C_{\beta\gamma\varepsilon}\widetilde{T}^{\varepsilon a}-C^{\ a}_{\beta\varepsilon}\widetilde{T}^{\varepsilon}{}_{\gamma}-C^{\ a}_{\gamma\varepsilon}\widetilde{T}^{\varepsilon}{}_{\beta},$$

where $\widetilde{T}^{\varepsilon a}=g^{a\beta}\widetilde{T}^{\varepsilon}{}_{\beta}$.

We shall call the above connection $\widetilde{\Gamma}=(\widetilde{\Gamma}^{\ a}_{\beta\gamma},\ \Gamma^{a}{}_{\gamma},\ C^{\ a}_{\beta\gamma}$(*resp.* (0)) on M_m the *semi-induced STD*(resp. *STD*(0))*-connection* and denote it by $\overset{s}{I}$*STDΓ*(resp. $\overset{s}{I}$*STD*Γ_o).

For the later use, we shall give the following relations :

$$(6.7)\qquad (C_{jkr}T^{ri})B^{ajk}_{i\beta\gamma}=C_{\beta\gamma\varepsilon}T^{\varepsilon a}+C_{\beta\gamma b}T^{ba},$$

(6 7) $\quad (C^{l}_{j\gamma} T^{r}_{k}) B^{\alpha jk}_{l\beta\gamma} = C^{\ \alpha}_{\beta\varepsilon} T^{\varepsilon}_{\ \gamma} + C^{\ \alpha}_{\beta b} T^{b}_{\ \gamma},$

(6. 8) $\quad \overline{C}_{\gamma} = C_{\gamma} - \sum_{b} C_{bb\gamma}, \ \overline{C}^{\gamma} = C^{\gamma} - \sum_{b} C^{\ \ \gamma}_{bb},$

where $T^{ba} = g^{\alpha\gamma} T^{b}_{\ \gamma}$, $\overline{C}_{\gamma} = C_{\alpha\beta\gamma} g^{\alpha\beta}$ and $C_{\gamma} = C_{k} B^{k}_{\ \gamma}$.

If we apply (5. 22) and (6. 3) to (6. 6) and make use of (6. 5) and (6. 7), than we obtain

$$\begin{aligned}\widetilde{\Gamma}^{\ \alpha}_{\beta\gamma} &= \Gamma^{\ \alpha}_{\beta\gamma} + C^{\ \alpha}_{\gamma b} H^{b}_{\ \beta} - C_{\beta\gamma b} H^{ba} \\ &= \overset{o}{\Gamma}{}^{\ \alpha}_{\beta\gamma} + C^{\ \alpha}_{\beta b} H^{b}_{\ \gamma} + C^{\ \alpha}_{\gamma b} H^{b}_{\ \beta} - C_{\beta\gamma b} H^{ba},\end{aligned} \tag{6. 9}$$

where $H^{ba} = H^{b}_{\ \varepsilon} g^{\varepsilon\alpha}$.

For the sake of brevity, we shall say that the *IMΓ* is simply *intrinsic* if it is the intrinsic *MΓ* on M_m. Further we shall say that the $\overset{s}{I}STD\Gamma$(resp. $\overset{s}{I}STD\Gamma$) is *intrinsic* if it is the intrinsic $STD\Gamma$(resp. $STD\Gamma_o$) on M_m.

Since the intrinsic $STD\Gamma$(or $STD\Gamma_o$) is an $STD\Gamma$(or $STD\Gamma_o$) on M_m determined by the tensor $T^{\alpha}_{\ \gamma}$, the *h*-connection $\overline{\Gamma}^{\ \alpha}_{\beta\gamma}$ of this connection is, because of (1. 22), given by

$$\overline{\Gamma}^{\ \alpha}_{\beta\gamma} = \Gamma^{*\ \alpha}_{\ \beta\gamma} + \overline{W}^{\ \alpha}_{\beta\gamma}, \quad \overline{W}^{\ \alpha}_{\beta\gamma} = C_{\beta\gamma\varepsilon} T^{\varepsilon\alpha} - C^{\ \alpha}_{\beta\varepsilon} T^{\varepsilon}_{\ \gamma} - C^{\ \alpha}_{\gamma\varepsilon} T^{\varepsilon}_{\ \beta}. \tag{6. 10}$$

Therefore it follows from (6. 6), (6. 10) and the first in (6. 3) that $\Gamma^{\alpha}_{\ \gamma} = G^{\alpha}_{\ \gamma} + T^{\alpha}_{\ \gamma}$ and $\widetilde{\Gamma}^{\ \alpha}_{\beta\gamma} = \overline{\Gamma}^{\ \alpha}_{\beta\gamma}$ if and only if $C^{\ \alpha}_{\beta b} \overset{b}{H}{}^{b}_{\ o} = 0$. From (6. 9) and the second in (6. 3) we have

$$\overline{Q}^{\ \alpha}_{\beta\gamma} = \widetilde{\Gamma}^{\ \alpha}_{\beta\gamma} - \Gamma^{\alpha}_{\ \gamma|\beta} = Q^{\ \alpha}_{\beta\gamma} + C^{\ \alpha}_{\gamma b} H^{b}_{\ \beta} - C^{\ \alpha}_{\beta b} H^{ba}, \tag{6. 11}$$

which implies that $\overline{Q}^{\ \alpha}_{\beta\gamma} = Q^{\ \alpha}_{\beta\gamma}$ if and only if $C^{\ \alpha}_{\beta b} H^{b}_{\ \gamma} = 0$.

Since the latter $C^{\ \alpha}_{\beta b} H^{b}_{\ \gamma} = 0$ implies the former $C^{\ \alpha}_{\beta b} \overset{b}{H}{}^{b}_{\ o} = 0$, we can state

Lemma 6.1. *The* $\overset{s}{I}STD\Gamma$*(or* $\overset{s}{I}STD\Gamma_o$*) is intrinsic if and only if an equation* $C^{\ \alpha}_{\gamma b} H^{b}_{\ \beta} = 0$ *holds.*

It follows from (6. 9) that $\widetilde{\Gamma}^{\ \alpha}_{\beta\gamma} = \Gamma^{\ \alpha}_{\beta\gamma}$ if and only if

$$C^{\ \ \gamma}_{\alpha b} H^{b}_{\ \beta} = C^{\ \ \gamma}_{\beta b} H^{b}_{\ \alpha}, \tag{6. 12}$$

and that $\widetilde{\Gamma}^{\ \alpha}_{\beta\gamma} = \overset{o}{\Gamma}{}^{\ \alpha}_{\beta\gamma}$ if and only if $C_{\beta\alpha b} H^{b}_{\ \gamma} = C_{\gamma\alpha b} H^{b}_{\ \beta} - C_{\beta\gamma b} H^{b}_{\ \alpha}$, which is equivalent to

$$C^{\ \alpha}_{\beta b} H^{b}_{\ \gamma} = 0. \tag{6. 13}$$

Hence, from Lemma 6. 1 and (6. 12) we can state

Theorem 6.3. *The $\overset{s}{I}STD\Gamma$ coincides with the $ISTD\Gamma$ if and only if the $ISTD\Gamma$ is h-symmetric, while the $\overset{s}{I}STD\Gamma_o$ coincides with the $ISTD\Gamma_o$ if and only if the $\overset{s}{I}STD\text{-}\Gamma_o$ is intrinsic. Any one of the connections $ISTD\Gamma$, $ISTD\Gamma_o$, $\overset{s}{I}STD\Gamma$ and $\overset{s}{I}STD\Gamma_o$ is intrisic if and only if the same condition* (6. 13) *holds.*

The induced connection $ISTD\Gamma$(resp. $ISTD\Gamma_o$) is called the *induced AMD* (resp. *AMD*(0))*-connection* and denoted by $IAMD\Gamma$(resp. $IAMD\Gamma_o$), if the original connection $STD\Gamma$(resp. $STD\Gamma_o$) is an $AMD\Gamma$(resp. $AMD\Gamma_o$) on M_n. Similarly the semi-induced AMD(resp. $AMD(0)$)-connection $\overset{s}{I}AMD\Gamma$(resp. $\overset{s}{I}AMD\Gamma_o$) can be defined. In this case, it follows from (1. 23), (6. 5), (6. 9), (6. 10) and (6. 13) that

$$(6.14)\qquad T^a{}_\gamma=\bar{f}\bar{L}h^a{}_\gamma,\ \ T^b{}_\gamma=0\,,$$

$$(6.15)\qquad \Gamma^a_\gamma=\overset{b}{\Gamma}{}^a_\gamma+\bar{f}\bar{L}h^a{}_\gamma,\ \ H^b_\gamma=\overset{b}{H}{}^b_\gamma,$$

$$(6.16)\qquad \overset{0}{\Gamma}{}^a_{\beta\gamma}=\overset{r}{\Gamma}{}^a_{\beta\gamma}-\bar{f}\bar{L}C^a_{\beta\gamma},\ \ \Gamma^a_{\beta\gamma}=\overset{c}{\Gamma}{}^a_{\beta\gamma}-\bar{f}\bar{L}C^a_{\beta\gamma},\ \ \bar{\Gamma}{}^a_{\beta\gamma}=\Gamma^{*a}_{\beta\gamma}-\bar{f}\bar{L}C^a_{\beta\gamma},$$

$$(6.17)\qquad \tilde{\Gamma}{}^a_{\beta\gamma}=\overset{r}{\Gamma}{}^a_{\beta\gamma}-\bar{f}\bar{L}C^a_{\beta\gamma}+C^a_{\beta b}\overset{b}{H}{}^b_\gamma+C^a_{\gamma b}\overset{b}{H}{}^b_\beta-C_{\beta\gamma b}\overset{b}{H}{}^{ba}.$$

Hence we can state

Corollary 6.3.1. *With respect to the connections $IAMD\Gamma$, $IAMD\Gamma_o$, $\overset{s}{I}AMD\Gamma$ and $\overset{s}{I}AMD\Gamma_o$, the following facts hold:*

(a) *The non-linear connection is commonly given by* (6. 15).

(b) *The h-connections are given by* (6. 16) *and* (6. 17), *provided that the third in* (6. 16) *is the intrinsic h-connection.*

(c) *Each connection is intrinsic if and only if* $C^a_{\beta b}\overset{b}{H}{}^b_\gamma=0$.

The induced connection $ISTD\Gamma$(resp. $ISTD\Gamma_o$) is called the *induced CD*(resp. *RD*)*-connection* and denoted by $ICD\Gamma$(resp. $IRD\Gamma$), if the original connection $S\text{-}TD\Gamma$(resp. $STD\Gamma_o$) is a $CD\Gamma$(resp. $RD\Gamma$) on M_n. Similarly the semi-induced CD (resp. RD)-connection $\overset{s}{I}CD\Gamma$(resp. $\overset{s}{I}RD\Gamma$) can be defined. From (1. 24) and (1. 25) we have

$$(6.18)\qquad T^a{}_\gamma=\bar{f}\bar{L}^3C^aC_\gamma,\ \ T^b{}_\gamma=\bar{f}\bar{L}^3C^bC_\gamma\ \ (C^b=C^iN^b_i),$$

$$(6.19)\qquad \Gamma^a_\gamma=\overset{b}{\Gamma}{}^a_\gamma+\bar{f}\bar{L}^3C^aC_\gamma,\ \ H^b_\gamma=\overset{b}{H}{}^b_\gamma+\bar{f}\bar{L}^3C^bC_\gamma.$$

By virtue of (6. 10) and (6. 18), the intrinsic h-connection of the four connections is commonly given by

$$\overline{\Gamma}^{\alpha}_{\beta\gamma}=\Gamma^{*\alpha}_{\beta\gamma}+\overline{W}^{\alpha}_{\beta\gamma},\ \overline{W}^{\alpha}_{\beta\gamma}=\overline{f}\overline{L}^{3}(C_{\beta\alpha\varepsilon}C^{\alpha}-C^{\alpha}_{\beta\varepsilon}C_{\gamma}-C^{\alpha}_{\gamma\varepsilon}C_{\beta})C^{\varepsilon}. \tag{6. 20}$$

From (6. 5) and (6. 9) we obtain

$$\begin{aligned}&\overset{\rho}{\Gamma}{}^{\alpha}_{\beta\gamma}=\overset{r}{\Gamma}{}^{\alpha}_{\beta\gamma}+W^{\alpha}_{\beta\gamma},\ \Gamma^{\alpha}_{\beta\gamma}=\overset{c}{\Gamma}{}^{\alpha}_{\beta\gamma}+\overline{W}^{\alpha}_{\beta\gamma}+\overline{f}\overline{L}^{3}(C_{\beta\gamma b}C^{\alpha}-C^{\alpha}_{\gamma b}C_{\beta})C^{b},\\ &W^{\alpha}_{\beta\gamma}=\overline{W}^{\alpha}_{\beta\gamma}+\overline{f}\overline{L}^{3}(C_{\beta\gamma b}C^{\alpha}-C^{\alpha}_{\beta b}C_{\gamma}-C^{\alpha}_{\gamma b}C_{\beta})C^{b},\end{aligned} \tag{6. 21}$$

$$\widetilde{\Gamma}^{\alpha}_{\beta\gamma}=\overset{r}{\Gamma}{}^{\alpha}_{\beta\gamma}+\overline{W}^{\alpha}_{\beta\gamma}+C^{\alpha}_{\beta b}\overset{b}{H}{}^{b}_{\gamma}+C^{\alpha}_{\gamma b}\overset{b}{H}{}^{b}_{\beta}-C_{\beta\gamma b}\overset{b}{H}{}^{b\alpha} \tag{6. 22}$$

Hence we can state

Corollary 6.3.2. *With respect to the connections $ICD\Gamma$, $IRD\Gamma$, $\overset{s}{I}CD\Gamma$ and $\overset{s}{I}RD\Gamma$, the following facts hold*:

(a) *The non-linear connection is commonly given by* (6. 19).

(b) *The h-connections are given by* (6. 21) *and* (2. 22), *while the intrinsic h-connection is given by* (6. 20).

(c) *Each connection is intrinsic if and only if* $C^{\alpha}_{\beta b}\overset{b}{H}{}^{b}_{\gamma}=-\overline{f}\overline{L}^{3}C^{\alpha}_{\beta b}C^{b}C_{\gamma}$.

Now we consider a tensor $\overline{T}^{\alpha}{}_{\gamma}$ on M_m defined by

$$\overline{T}^{\alpha}{}_{\gamma}=\overline{f}\overline{L}^{3}\overline{C}^{\alpha}\overline{C}_{\gamma}. \tag{6. 23}$$

Then from (6. 8) we have $\overline{T}^{o}_{\gamma}=\overline{T}^{\alpha}{}_{o}=0$. Therefore an $STD\Gamma$(resp. $STD\Gamma_{o}$) on M_m is uniquely determined by this tensor $\overline{T}^{\alpha}{}_{\gamma}$. This connection is defined as follows :

$$\begin{aligned}&\Gamma^{\alpha}{}_{\gamma}=G^{\alpha}{}_{\gamma}+\overline{f}\overline{L}^{3}\overline{C}^{\alpha}\overline{C}_{\gamma},\ \widetilde{C}^{\alpha}_{\beta\gamma}=C^{\alpha}_{\beta\gamma}\quad (resp.\ \widetilde{C}^{\alpha}_{\beta\gamma}=0),\\ &\Gamma^{\alpha}_{\beta\gamma}=\Gamma^{*\alpha}_{\beta\gamma}+\overline{f}\overline{L}^{3}(C_{\beta\gamma\varepsilon}\overline{C}^{\alpha}-C^{\alpha}_{\beta\varepsilon}\overline{C}_{\gamma}-C^{\alpha}_{\gamma\varepsilon}\overline{C}_{\beta})\overline{C}^{\varepsilon}.\end{aligned} \tag{6. 24}$$

We shall call the above connection the *naturally intrinsic* CD(resp. RD)*-connection.* From (6. 8) and Corollary 3.3.2 we can state

Corollary 6.3.3. *The $ICD\Gamma$(resp. $IRD\Gamma$) is the naturally intrinsic CD(resp. RD)-connection if and only if the following equations hold*:

$$C^{\alpha}_{\beta b}\overset{b}{H}{}^{b}_{\gamma}=-\overline{f}\overline{L}^{3}C^{\alpha}_{\beta b}C^{b}C_{\gamma},\quad \delta^{bc}C_{bc\gamma}=0. \tag{6. 25}$$

The above fact is also valid for the $\overset{s}{I}CD\Gamma$(resp. $\overset{s}{I}RD\Gamma$).

The induced connection $ITMD\Gamma$(resp. $ITMD\Gamma_o$) is called the *GQD*(resp. *GQD(0))-connection* and denoted by $IGQD\Gamma$(resp. $IGQD\Gamma_o$), if the original connection $TMD\Gamma$(resp. $TMD\Gamma_o$) is a $GQD\Gamma$(resp. $GQD\Gamma_o$). In this case, we have $T^a{}_\gamma=0$ and $H^b_\gamma=\overset{b}{H}{}^b_\gamma$. Therefore from (6. 1) and (6. 3) we have

$$\widetilde{T}^a_\gamma=-C^a_{\gamma b}\overset{b}{H}{}^b_\beta,\quad \widetilde{C}^a_{\beta\gamma}=C^a_{\beta\gamma}\quad (resp.\ \widetilde{C}^a_{\beta\gamma}=0),$$
(6. 26)
$$\widetilde{Q}^a_{\beta\gamma}=Q^a_{\beta\gamma}-C^a_{\beta b}\overset{b}{H}{}^b_\gamma\quad (resp.\ \widetilde{Q}^a_{\beta\gamma}=Q^a_{\beta\gamma}-2C^a_{\beta b}\overset{b}{H}{}^b_\gamma).$$

From (1. 28) we obtain

$$\Gamma^a_\gamma=\overset{b}{\Gamma}{}^a_\gamma=G^a_\gamma+\widetilde{T}^a_\gamma,\quad W^a_{\beta\gamma}=Q^a_{\beta\gamma}+P^a_{\beta\gamma},$$
(6. 27)
$$\overset{o}{\Gamma}{}^a_{\beta\gamma}=\overset{r}{\Gamma}{}^a_{\beta\gamma}+W^a_{\beta\gamma},\quad \Gamma^a_{\beta\gamma}=\overset{c}{\Gamma}{}^a_{\beta\gamma}+W^a_{\beta\gamma}.$$

Hence we can state

Theorem 6.4. *The $IGQD\Gamma$(resp. $IGQD\Gamma_o$) is a $TMD\Gamma$(resp. $TMD\Gamma_o$) on M_m determined by tensors $\widetilde{T}^a_\gamma$, $\widetilde{Q}^a_{\beta\gamma}$ and $\widetilde{C}^a_{\beta\gamma}$ in* (6. 26), *and the non-linear connection and the h-connection are given by* (6. 27). *Each connection is intrinsic if and only if $C^a_{\beta b}\overset{b}{H}{}^b_\gamma=0$.*

The $IGQD\Gamma$(resp. $IGQD\Gamma_o$) is called the *induced MD*(resp. *MD(0))-connection* and denoted by $IMD\Gamma$(resp. $IMD\Gamma_o$), if the original connection $GQD\Gamma$(resp. $GQD\Gamma_o$) is an $MD\Gamma$(resp. $MD\Gamma_o$). From (1. 29) and (6. 27) we have

(6. 28) $$\overset{o}{\Gamma}{}^a_{\beta\gamma}=\overset{r}{\Gamma}{}^a_{\beta\gamma}+\overline{f}(l_\beta\delta^a_\gamma-l^a g_{\beta\gamma}),\quad \Gamma^a_{\beta\gamma}=\overset{o}{\Gamma}{}^a_{\beta\gamma}+C^a_{\beta b}\overset{b}{H}{}^b_\gamma.$$

Hence we can state

Corollary 6.4.1. *With respect to the connections $IMD\Gamma_o$ and $IMD\Gamma$, the following facts hold:*

(a) *The h-connections are given by* (6. 28).

(b) *The $IMD\Gamma$ is h-metrical.*

The $IGQD\Gamma$(resp. $IGQD\Gamma_o$) is called the *induced $\widetilde{H}D$*(resp. *$\widetilde{B}D$)-connection* and denoted by $I\widetilde{H}D\Gamma$(resp. $I\widetilde{B}D\Gamma$), if the original connection $GQD\Gamma$(resp. $GQD\Gamma_o$) is a $\widetilde{H}D\Gamma$(resp. $\widetilde{B}D\Gamma$). Then we can state

Corollary 6.4.2. *With respect to the connections $I\widetilde{H}D\Gamma$ and $I\widetilde{B}D\Gamma$, the following facts hold:*

(a) *The* $\widetilde{IHD\Gamma}$ *is h-symmetric if and only if* $C^{\ a}_{\beta b}\overset{b}{H}{}^{b}_{\ o}=0$.

(b) *The* $\widetilde{IBD\Gamma}$ *is h-symmetric.*

The $\widetilde{IHD\Gamma}$(resp. $\widetilde{IBD\Gamma}$) is called the *induced HD*(resp. *BD*)*-connection* and denoted by $IHD\Gamma$(resp. $IBD\Gamma$), if the original connection $\widetilde{HD\Gamma}$(resp. $\widetilde{BD\Gamma}$) is a $HD\Gamma$(resp. $BD\Gamma$). Then, because of (1. 30) and (6. 27) we can state

Corollary 6 .4 .3 . *For the connections IHDΓ and IBDΓ, the h-connections are given by*

$$\Gamma^{\alpha}_{\beta\gamma}=\overset{h}{\Gamma}{}^{\alpha}_{\beta\gamma}+Q^{\ \alpha}_{\beta\gamma},\quad \overset{o}{\Gamma}{}^{\alpha}_{\beta\gamma}=\overset{b}{\Gamma}{}^{\alpha}_{\beta\gamma}+Q^{\ \alpha}_{\beta\gamma},\tag{6. 29}$$

where $Q^{\ \alpha}_{\beta\gamma}=\bar{f}\,\bar{L}^{2}(l_{\beta}C^{\alpha}C_{\gamma}+l_{\gamma}C^{\alpha}C_{\beta}-l^{\alpha}C_{\beta}C_{\gamma})$.

The $\widetilde{IHD\Gamma}$(resp. $\widetilde{IBD\Gamma}$) is called the *induced AMBD*(resp. *AMBD*(0))*-connection* and denoted by $IAMBD\Gamma$(resp. $IAMBD\Gamma_{o}$), if the original connection is an $AMBD\Gamma$(resp. $AMBD\Gamma_{o}$). The above definition and notation will be applied to also an $AMCD\Gamma$(resp. $AMCD\Gamma_{o}$).

From (1. 31), (1. 32) and (6. 27) we can state

Corollary 6 .4 .4 . *For the connections IAMBDΓ(resp. IAMBDΓ_{o}) and IAMC-DΓ(resp. IAMCDΓ_{o}), the h-connections are respectively given by*

$$\Gamma^{\alpha}_{\beta\gamma}=\overset{h}{\Gamma}{}^{\alpha}_{\beta\gamma}+Q^{\ \alpha}_{\beta\gamma},\quad \overset{o}{\Gamma}{}^{\alpha}_{\beta\gamma}=\overset{b}{\Gamma}{}^{\alpha}_{\beta\gamma}+Q^{\ \alpha}_{\beta\gamma},\tag{6. 30}$$

$$\Gamma^{\alpha}_{\beta\gamma}=\overset{c}{\Gamma}{}^{\alpha}_{\beta\gamma}+Q^{\ \alpha}_{\beta\gamma},\quad \overset{o}{\Gamma}{}^{\alpha}_{\beta\gamma}=\overset{r}{\Gamma}{}^{\alpha}_{\beta\gamma}+Q^{\ \alpha}_{\beta\gamma},\tag{6. 31}$$

where $Q^{\ \alpha}_{\beta\gamma}=\bar{f}(l_{\beta}h^{\alpha}{}_{\gamma}+l_{\gamma}h^{\alpha}{}_{\beta}-l^{\alpha}h_{\beta\gamma})$.

§ 7 . Special subspaces. In this section we shall be concerned with various special subspaces. Let us first consider a curve $C: u^{\alpha}=u^{\alpha}(s)$ (s: arc-length) in M_{m}. Since $x^{i}=x^{i}(u^{\alpha}(s))$ along C, the unit tangent vector is given by $dx^{i}/ds=l^{i}=B^{i}{}_{\alpha}(du^{\alpha}/ds)=B^{i}{}_{\alpha}l^{\alpha}$. Then from (3. 49) we have

$$Dl^{i}/ds=B^{i}{}_{\gamma}(dl^{\gamma}/ds)+N^{i}{}_{a}(H^{\ a}_{\gamma}du^{\gamma}/ds).\tag{7. 1}$$

Since $Dl^{i}/ds=0$ and $Dl^{\gamma}/ds=0$ are equations of paths with respect to $M\Gamma$ and $IM\Gamma$ respectively, it follows from (7. 1) that each path in M_{m} is a path in M_{n} if and only if $H^{\ a}_{\gamma}=0$ ($a=m+1,\ \ldots,\ n$).

We shall say that M_{m} is a *totally auto-parallele subspace* (or simply *totally auto-parallel*) *with respect to IMΓ* if each path in M_{m} with respect to $IM\Gamma$ is also a path in M_n with respect to $M\Gamma$.

Then we can state

Teorem 7.1. *With respect to the induced connection IMΓ, the following facts are mutually equivalent*:

(1) *A subspace M_m of M_n is totally auto-parallel.*

(2) *The normal curvature vector H^a_γ in each normal direction N^a_i vanishes identically.*

(3) *The pair (Γ^i_k, Γ^a_β) has the (H)-property, namely*

$$B^i_{0\gamma} - \Gamma^a_\gamma B^i_a + \Gamma^i_k B^k_\gamma = 0 \qquad ([21],\ [34]). \tag{7.2}$$

(4) *The two tensors $\overset{b}{H}{}^a_{\beta\gamma}(=N^a_i(B^i_{\beta\gamma}+G^i_{jk}B^{jk}_{\beta\gamma}))$ and $T^a_{\beta\gamma}(=N^a_i T^i_{jk} B^{jk}_{\beta\gamma})$ are related by*

$$\overset{b}{H}{}^a_{\beta\gamma} + T^a_{\beta\gamma} = 0. \tag{7.3}$$

Proof. It is clear that two facts (1) and (2) are mutually equivalent. If $H^a_\gamma = 0$, then from (3. 22) we have (7. 2). Conversely if we apply (7. 2) to (3. 13), then we obtain $H^a_\gamma = 0$. Next, if we differentiate (3. 13) by y^β, then from (1. 3), (2. 22), (3. 12), (3. 14) and (3. 15) we have

$$H^a_{\gamma|\beta} = (\lambda^a_{b\beta} - \widetilde{C}^a_{\beta b})H^b_\gamma + H^a_{\beta\gamma} - Q^a_{\beta\gamma} \quad (Q^a_{\beta\gamma} = N^a_i Q^i_{jk} B^{jk}_{\beta\gamma}). \tag{7.4}$$

On the other hand, the tensor $H^a_{\beta\gamma}$ is expressible in

$$H^a_{\beta\gamma} = \overset{b}{H}{}^a_{\beta\gamma} + T^a_{\beta\gamma} + Q^a_{\beta\gamma} + \widetilde{C}^a_{\beta b} H^b_\gamma. \tag{7.5}$$

If $H^a_\gamma = 0$, then it follows from (7. 4) and (7. 5) that the condition (7. 3) is satisfied. Conversely, contraction of (7. 3) by y^β yeilds

$$\overset{b}{H}{}^a_\gamma + T^a_\gamma = H^a_\gamma = 0. \qquad \text{Q. E. D.}$$

Note 7.1. In the above Theorem, the second fact $H^a_\gamma = 0$ implies $H^a_{\beta\gamma} = Q^a_{\beta\gamma}$ because of (7. 4). However the latter does, in general, not imply the former.

Next we can state

Theorem 7.2. *Let M_n be endowed with a geo-path connection $M\Gamma$. Then the induced connection IMΓ is also a geo-path connection on M_m. With respect to this IMΓ, the following facts are mutually equivalent*:

(1) *The subspace M_m of M_n is totally geodesic.*

(2) $H^a_\gamma = 0$ $(a = m+1, \ldots, n)$.

(3) *The pair* $(\Gamma^i_k, \Gamma^a_\gamma)$ *has the (H)-property.*

(4) $\overset{b}{H}{}^a_{\beta\gamma} = T^a_{\beta\gamma} = 0$ $(a = m+1, \ldots, n)$.

(5) *The second fundamental tensor* $H^a_{\beta\gamma}$ *in each direction* N^a_i *is given by*

$$H^a_{\beta\gamma} = Q^a_{\beta\gamma}, \tag{7.6}$$

provided $\det(\delta^a_b + \widetilde{C}^a_{ob}) \neq 0$.

Proof. Taking account of Lemma 3. 5, we can prove in the same way as before that the facts (1), (2) and (3) are mutually equivalent. Since the $M\Gamma$ is a geo-path connection, we have $T^i{}_o = 0$ and hence $T^a{}_o = 0$. Therefore $H^a_\gamma = 0$ implies $\overset{b}{H}{}^a_o = 0$, differentiation of which by y^γ yields $\overset{b}{H}{}^a_\gamma = 0$. Further, by differentiating this by y^β we have $\overset{b}{H}{}^a_{\beta\gamma} = 0$. Hence from (7. 3) we get $T^a_{\beta\gamma} = 0$. After all we have (4). Conversely, contraction of (4) by y^β yields $\overset{b}{H}{}^a_\gamma = T^a{}_\gamma = 0$ and hence $H^a_\gamma = 0$. Lastly if (2) holds, then from (7. 5) and (4) we have (7. 6). Conversely, suppose that (7. 6) holds. Then from (7. 5) we obtain

$$\overset{b}{H}{}^a_{\beta\gamma} + T^a_{\beta\gamma} + \widetilde{C}^a_{\beta b} H^b_\gamma = 0, \tag{7.7}$$

contraction of which by $y^\beta y^\gamma$ yields $\overset{b}{H}{}^a_o + \widetilde{C}^a_{ob} \overset{b}{H}{}^b_o = 0$. Since $\det(\delta^a{}_b + \widetilde{C}^a_{ob}) \neq 0$, this equation implies $\overset{b}{H}{}^a_o = 0$. Hence as before we have $\overset{b}{H}{}^a_\gamma = 0$. Therefore, contraction of (7. 7) by y^γ yields $T^a_{\beta\gamma} y^\gamma = 0$. On the other hand, if we differentiate $T^i{}_k y^k = 0$ by y^j then we have $T^i_{jk} y^k + T^i{}_j = 0$, which implies $T^a_{\beta\gamma} y^\gamma + T^a{}_\beta = 0$. Therefore we get $T^a{}_\beta = 0$. Consequently we obtain $H^a_\beta = \overset{b}{H}{}^a_\beta + T^a{}_\beta = 0$. Q. E. D.

From Theorem 3. 2 and Theorem 7. 2 we can state

Corollary 7.2.1. *Let M_n be endowed with a geo-path connection $M\Gamma$. Then any one of the five facts* (1) ~ (5) *in Theorem* 7. 2 *is a sufficient condition for the induced connection $IM\Gamma$ to be intrinsic.*

Proof. The second fact $H^b_\gamma = 0$ implies $\overset{b}{H}{}^b_o = 0$. Therefore the condition $(3.43)_3$ holds. Hence the $IM\Gamma$ is intrinsic. Q. E. D.

Note 7.2. Connections $TM\Gamma$, $TM\Gamma_o$, $TMD\Gamma$ and $TMD\Gamma_o$ are all geo-path connections. Therefore, for the induced connections on M_m from the above connections, Theorem 7. 2 and Corollary 7. 2. 1 are valid. In these cases, we have always $\det(\delta^a{}_b + \widetilde{C}^a_{ob}) = \det(\delta^a{}_b) = 1$.

Next we shall consider a curve $C: u^a = u^a(s)$ in M_m endowed with a vector fields $y^a(s)$. Since $x^i = x^i(u^a(s))$ along C, we have $y^i = B^i_a y^a(s)$ and $dx^i/ds = B^i_a du^a/ds$. Then it follows from (3. 49) and (3. 50) that

$$Dy^i/ds = B^i_a (Dy^\gamma/ds) + N^i_a (H^a_\gamma du^\gamma/ds),$$

(7. 8)
$$D(dx^i/ds) = B^i_\gamma \{D(du^\gamma/ds)/ds\} + N^i_a \{H^a_{\beta\gamma}(du^\beta/ds)(du^\gamma/ds) + \widetilde{C}^a_{\beta\gamma}(du^\beta/ds)(Dy^\gamma/ds)\}.$$

The curve C is called an *h-path in* M_m *with respect to IMΓ* if it satisfies $Dy^a/ds = 0$ and $D(du^a/ds)/ds = 0$, while the C is called an *h-path in* M_n *with respect to MΓ* if it satisfies $Dy^i/ds = 0$ and $D(dx^i/ds)/ds = 0$. We shall say that M_m is *a totally h-auto-parallel subspace* (or simply *totally h-auto-parallel*) *with respect to IMΓ* if each h-path in M_m with respect to *IMΓ* is always an h-path in M_n with respect to *MΓ*. From (7. 8) we can state

Lemma 7.1. *M_m is totally h-auto-parallel with respect to IMΓ if and only if the following equations hold:*

$$H^a_\gamma = 0,\ ½(H^a_{\beta\gamma} + H^a_{\gamma\beta}) = 0\ (a = m+1, \ldots\ldots, n).$$

From Theorem 7. 1, Lemma 7. 1 and Note 7. 1 we can state

Theorem 7.3. *With respect to the induced connection IMΓ, M_m is totally h-auto-parallel if and only if M_m is totally auto-parallel and the tensors $Q_\beta{}^a{}_\gamma$ $(a = m+1, \ldots\ldots, n)$ are skew-symmetric in β and γ.*

By virtue of Theorem 7. 2 and Theorem 7. 3 we can state

Corollary 7.3.1. *Let M_n be endowed with a geo-path connection MΓ. Then with respect to IMΓ, M_m is totally h-auto-parallel if and only if M_m is totally geodesic and the tensors $Q^a_{\beta\gamma}$ $(a = m+1, \ldots\ldots, n)$ are skew-symmetric in β and γ.*

Note 7.3. Let M_n be endowed with an *MΓ* satisfying $Q_j{}^i{}_k = 0$ or $Q_j{}^i{}_k + Q_k{}^i{}_j = 0$. Then with respect to *IMΓ*, the following facts are the same : (a) M_m is totally h-auto-parallel. (b) M_m is totally auto-parallel.

Further if the *MΓ* is a geo-path connection, then the facts (a), (b) and the following fact (c) are the same : (c) M_m is totally geodesic. As the practical examples of the latter, we have *BΓ*, *HΓ* and *ISΓ* etc.

We put

$$(7.9)\qquad \widetilde{R}(x, y, X)=\frac{\widetilde{R}_{jikh}y^j y^k X^i X^h}{(g_{jk}g_{ih}-g_{jh}g_{ik})y^j y^k X^i X^h},$$

where $\widetilde{R}_{jikh}$ is the h-curvature tensor with respect to $M\Gamma$. Then we shall call $\widetilde{R}(x, y, X)$ the *sectional curvature defined by* y^i *and* X^i *with respect to* $M\Gamma$. Further we shall say that M_n is *of scalar curvature* $\widetilde{R}$ or *of constant curvature* $\widetilde{R}$ *with respect to* $M\Gamma$ according to whether the sectional curvature is independent of X^i or it is constant.

Similarly we shall apply the above definitions to M_m with respect to $IM\Gamma$. If we contract (a) in (4. 12) by $y^\alpha y^\beta$, then we have

$$(7.10)\qquad \begin{aligned}\widetilde{R}_{0\sigma 0\gamma}&=\widetilde{R}_{0i0h}B^{ih}_{\sigma\gamma}+\widetilde{S}_{0ikh}B^i_\sigma N^k_a N^h_b H^a_0 H^b_\gamma\\ &+B^i_\sigma N^h_a(\widetilde{P}_{0i0h}H^a_\gamma-\widetilde{P}_{0ikh}B^k_\gamma H^a_0)+H^a_{00}(g_{jk|\gamma}B^j_\sigma N^k_a+\delta_{ab}H^b_{\sigma\gamma})\\ &-H^a_{0\gamma}(g_{jk|\beta}y^\beta B^i_\sigma+\delta_{ab}H^b_{\sigma 0}).\end{aligned}$$

It follows from (7. 9) and the definition that M_n is of scalar (resp. constant) curvature $\widetilde{R}$ with respect to $M\Gamma$ if and only if the following holds :

$$(7.11)\qquad \tfrac{1}{2}(\widetilde{R}_{0i0h}+\widetilde{R}_{0h0i})=L^2\widetilde{R}h_{ih}\ (resp.\ \widetilde{R}: constant).$$

Suppose that $H^a_\gamma=0$. Then from Note 7. 1 we have $H^a_{0\gamma}=Q^a_{0\gamma}=D^i{}_k N^a_i B^k_\gamma=D^a{}_\gamma$. Therefore we shall say that M_m is *totally auto-parallel with vanishing* D *with respect to* $IM\Gamma$ if the second fact $H^a_\gamma=0$ $(a=m+1, \ldots, n)$ implies $D^a{}_\gamma=0$.

In this case, if (7. 11) holds then from (7. 10) we obtain

$$(7.11)_1\qquad \tfrac{1}{2}(\widetilde{R}_{0\sigma 0\gamma}+\widetilde{R}_{0\gamma 0\sigma})=\overline{L}^2\widetilde{R}h_{\sigma\gamma}.$$

In view of $(7.11)_1$ we can state

Teorem 7.4. *If* M_n *is of scalar (resp. constant) curvature* $\widetilde{R}$ *with respect to* $M\Gamma$ *and* M_m *is totally auto-parallel with vanishing* D *with respect to* $IM\Gamma$, *then* M_m *is also of scalar (resp. constant) curvature* $\widetilde{R}$ *with respect to* $IM\Gamma$.

Further we can state

Corollary 7.4.1. *Let* Γ *be any one of the following connections* :

$T M\Gamma$, $AMD\Gamma$, $MD\Gamma$, $AMBD\Gamma$, $AMCD\Gamma$, *and the five corresponding connections with* Γ_o. *Then if* M_n *is of scalar (resp. constant) curvature* $\widetilde{R}$ *with respect to* Γ *and* M_m *is totally geodesic, then* M_m *is also of scalar (resp. constant) curvature* $\widetilde{R}$ *with respect to the induced connection* $I\Gamma$.

Proof. A TM (or $TM\Gamma_o$) is dft-free, namely $D^i_{\ k}=0$. For the remaining connections, we have $D^i_{\ k}=fLh^i_{\ k}$ (or $-fLh^i_{\ k}$). Therefore, in any case we have $D^a_{\ \gamma}=0$. The above connections are all geo-path connections. Q. E. D.

It is known [28] that the Weyl tensor $W^i_{\ kh}$ vanishes if and only if M_n is of scalar curvature with respect to $B\Gamma$. We shall say that M_n is *W-flat* if the Weyl tensor vanishes.

$B\Gamma$ is a special $TM\Gamma_o$ and hence a geo-path connection. Therefore by virtue of Corollary 7. 2. 1 and Corollary 7. 4. 1 we can state

Corollary 7.4.2. *If* M_n *is W-flat and* M_m *is totally geodesic, then* M_m *is also W-flat.*

We shall say that M_n is *D-flat* if the Douglas tensor $D_j{}^i{}_{kh}$ vanishes. Then we can state

Lemma 7.2. *If* M_n *is D-flat and* M_m $(n>m>2)$ *is totally geodesic, then* M_m *is also D-flat.*

Proof. From D-flatness on M_n, the hv-curvature tensor $G_j{}^i{}_{kh}$ with respect to $B\Gamma$ is expressible in

$$G^i_{jkh}=(y^i G_{jk\|h}+\delta^i_j G_{kh}+\delta^i_k G_{hk}+\delta^i_h G_{jk})/(n+1), \tag{7.12}$$

where $G^i_{jkh}=G^i_{jk\|h}$ and $G_{jk}=G^i_{jki}$.

Since M_m is totally geodesic, the induced connection $IB\Gamma$ is intrinsic. Therefore the h-connection is given by

$$G^a_{\beta\gamma}=B^a_{\ i}(B^i_{\beta\gamma}+G^i_{jk}B^{jk}_{\beta\gamma}). \tag{7.13}$$

If we differentiate (7. 13) by y^σ and take account of $\overset{b}{H}{}^b_{\beta\gamma}=0$, then we have

$$G^a_{\beta\gamma\sigma}=G^a_{\beta\gamma\|\sigma}=2C^a_{\sigma b}\overset{b}{H}{}^b_{\beta\gamma}+G^i_{jkh}B^{ajkh}_{i\beta\gamma\sigma}=G^i_{jkh}B^{ajkh}_{i\beta\gamma\sigma}. \tag{7.14}$$

Substituting (7. 12) into (7. 14), we obtain

$$G^a_{\beta\gamma\sigma}=(y^a G_{\beta\gamma\|\sigma}+\delta^a_\beta G_{\gamma\sigma}+\delta^a_\gamma G_{\sigma\beta}+\delta^a_\sigma G_{\beta\gamma})/(n+1), \tag{7.15}$$

where $G_{\beta\gamma}=G_{jk}B^{jk}_{\beta\gamma}$. The tensor $G_{\beta\gamma}$ is $(-1)p$-homogeneous in y^a. Taking account of this fact and contracting (7. 15) in α and δ, we have

$$\overline{G}_{\beta\gamma} = G^{\alpha}_{\beta\gamma\alpha} = (m+1)G_{\beta\gamma}/(n+1). \tag{7.16}$$

Applying (7. 16) to (7. 15), we obtain

$$G^{\alpha}_{\beta\gamma\sigma} = (y^{\alpha}\overline{G}_{\beta\gamma|\sigma} + \delta^{\alpha}_{\beta}\overline{G}_{\gamma\sigma} + \delta^{\alpha}_{\gamma}\overline{G}_{\sigma\beta} + \delta^{\alpha}_{\sigma}\overline{G}_{\beta\sigma})/(m+1),$$

which shows that M_m is also *W-flal.* Q. E. D.

It is said that M_n is projectively flat if M_n is both *W*-flat and *D*-flat. Then, from Corollary 7. 4. 2 and Lemma 7. 2 we can state

Theorem 7.5. *If M_n is projectively flat and $M_m(n>m>2)$ is totally geodesic, then M_m is also projectively flat.*

Let us now return to (7. 1). We shall call Dl^i/ds, $B^i_{\ \gamma}(Dl^{\gamma}/ds)$ and $N^i_{\ a}(H^a_{\ \gamma}du^{\gamma}/ds)$ the absolute curvature vector, the relative curvature vector and the normal vector, of the curve C, respectively.

If $du^{\alpha} = y^{\alpha}(ds/\overline{L})$, then the following relation holds :

$$H^a_{\ \gamma}(u^{\alpha},\ du^{\alpha}/ds)du^{\gamma}/ds = \frac{H^a_{\ 0}(u^{\alpha},\ du^{\alpha})}{\overline{L}^2(u^{\alpha},\ du^{\alpha})} = \frac{H^a_{\ 0}(u^{\alpha},\ y^{\alpha})}{\overline{L}^2(u^{\alpha},\ y^{\alpha})}. \tag{7.17$_1$}$$

In particular if $H^a_{\ 0\gamma} = H^a_{\ \gamma}$, then (7. 17)$_1$ is also expressible in

$$H^a_{\ \gamma}du^{\gamma}/ds = \frac{H^a_{\ \beta\gamma}(u^{\alpha},\ du^{\alpha})du^{\beta}du^{\gamma}}{g_{\beta\gamma}(u^{\alpha},\ du^{\alpha})du^{\beta}du^{\gamma}} = \frac{H^a_{\ 0}(u^{\alpha},\ y^{\alpha})}{\overline{L}^2(u^{\alpha},\ y^{\alpha})}. \tag{7.17$_2$}$$

In view of (7. 17)$_1$ and (7. 17)$_2$ we put

$$N^a(u^{\alpha},\ y^{\alpha}) = H^a_{\ 0}(u^{\alpha},\ y^{\alpha})/\overline{L}^2(u^{\alpha},\ y^{\alpha}). \tag{7.18}$$

In this case, the square of the normal curvature $N(u^{\alpha},\ y^{\alpha})$ in y^{α}-direction at a point (u^{α}) of M_m is given by

$$N^2(u^{\alpha},\ y^{\alpha}) = \delta_{ab}N^aN^b. \tag{7.19}$$

Therefore N^a is the $N^i_{\ a}$-component of N. This is the reason why $H^a_{\ \gamma}$ is called the normal curvature vector in a direction $N^i_{\ a}$.

In the following, we assume that M_n is endowed with a geo-path connection. Then we have

(7. 20) $\quad T^{a}{}_{o}=0,\ H^{a}_{o}=\overset{b}{H}{}^{a}_{o}=N^{a}_{i}(B^{i}_{oo}+2G^{i}).$

If we put

(7. 21) $\quad \Omega^{a}_{\beta\gamma}=\frac{1}{2}H^{a}_{0\|\gamma\|\beta},$

then from (2. 22) and (7. 2) we have

(7. 22) $\quad \Omega^{a}_{\beta\gamma}=\overset{b}{H}{}^{a}_{\beta\gamma}+(\lambda^{a}_{b\gamma}\overset{b}{H}{}^{b}_{\gamma}+\lambda^{a}_{b\gamma}\overset{b}{H}{}^{b}_{\beta})+\frac{1}{2}\overset{b}{H}{}^{c}_{o}(\lambda^{a}_{c\gamma\|\beta}+\lambda^{a}_{b\gamma}\lambda^{b}_{c\beta}).$

We shall call $\Omega^{a}_{\beta\gamma}$ the *second canonical fundamental tensor in a direction* N^{i}_{a}. Next we shall say that a scalar $C(u^{\alpha}, y^{\alpha})$ on M_{m} is *direct-free* if it is independent of y^{α}, namely $C_{\|\alpha}=0$.

We shall call a point (u^{α}) of M_{m} an *ncd-free point* or an *nc-constant* point according to whether the normal curvature $N(u^{\alpha}, y^{\alpha})$ at the point (u^{α}) is direct-free or constant. Further we shall say that M_{m} is *totally ncd-free*(resp. *nc-constant*) if every point of M_{m} is an ncd-free (resp. nc-constant) point.

Note 7.4. An ncd-free (resp. nc-constant) point corresponds to an umbilical (resp. a proper umbilical) point in Riemannian geometry.

Let M_{m} be totally ncd-free (resp. nc-constant). Then for direct-free scalars C^{a} (resp. constant C^{a}), we can put

(7. 23) $\quad H^{a}_{o}=\bar{L}^{2}C^{a}\quad (a=m+1, \ldots, n)$

Differentiating (7. 23) two times by y^{α} and y^{β} and making use of (7. 21), we have

(7. 24) $\quad \Omega^{a}_{\beta\gamma}=C^{a}g_{\beta\gamma}\quad (a=m+1, \ldots, n).$

Conversely let (7. 24) be satisfied. Then, contracting (7. 24) by $y^{\beta}y^{\gamma}$ and using (7. 21), we have (7. 23). Therefore from (7. 18) and (7. 19) we obtain $N^{2}=\delta_{ab}C^{a}C^{b}$. Consequently we can state

Theorem 7.6. *A subspace M_{m} of M_{n} is totally ncd-free(resp. nc-constant) if and only if the second canonical fundamental tensors $\Omega^{a}_{\beta\gamma}(a=m+1, \ldots, n)$ are expressible in* (7. 24) *for direct-free scalars* (*resp. constants*) C^{a}.

Note 7.5. If each scalar C^{a} vanishes in the above Theroem, then we have $\overset{b}{H}{}^{a}_{\gamma}=0$. Therefore M_{m} is totally geodesic if $T^{a}{}_{\gamma}=0$.

REFERENCES

[1] W. Barthel : Über die Minimalflächen in gefaserten Finslerräumen, *Annali di Mat,* **36** (1954), 159–190.

[2] A. Bejuncu : Geometry of Finsler subspaces I, II, to appear.

[3] L. Berwald : Über Parallelübertragung in Räumen mit allgemeiner Massbestimmung, *Jber. Deutsch. Math.-Verein,* **34** (1926), 213–220.

[4] L. Berwald : Über die Hauptkrummungen einer Fläche im dreidimensionalen Finslerschen Raum, *Monatsh, Math. Phys,* **43** (1936), 39–58.

[5] E. Cartan : Les espaces de Finsler, *Actualités* **79**, *Paris,* 1934.

[6] E. T. Davis : Subspaces of a Finsler space, *Proc. London Math. Soc.,* (2) **49** (1947), 19–39.

[7] M. Haimovici : Formules fondamentales dans la théorie des hypersurfaces d'un espace de Finsler, *C. R. Acad. Soc. Paris,* **198** (1934), 426–427.

[8] M. Haimovici : Variétés totalement extrêmals et variétés totalment géodésique dans les e-spaces de Finsler, *An. Sci. Univ. Iassy,* **25** (1939), 559–644.

[9] M. Hashiguchi : On parallel displacements in Finsler spaces, *J. Math. Soc. Japan* **10** (1958), 365–379.

[10] M. Hashiguchi : On determinations of Finsler connections by deflection tensor fields, *Rep. Fac. Sci. Kagoshima Univ.* **2** (1969), 29–39.

[11] M. Hashiguchi : Wagner connections and Miron connections of Finsler spaces, *Rev. Roum. Math. Pures et Appl.,* **25** (1980), 1387–1390.

[12] H. Hombu : Die Krümmungstheorie in Finslerschen Räume, *J. Fac. Sci. Hokkaido,* **5** (19-36), 67–94.

[13] A. Kawaguchi : On the theory of non-linear connections II, Theory of Minkowski spaces and non-linear connections in a Finsler space, *Tensor, N. S.,* **6** (1956), 165–199.

[14] S. Kikuchi : On the theory of subspaces in a Finsler space, *Tensor, N. S.,* **2** (1952), 67–79.

[15] D. Laugwitz : Zur geometrischen Begründung der Parallelverschiebung in Finslerschen Räumen, *Arch. Math.,* **6** (1955), 448–453.

[16] M. Matsumoto : The induced and intrinsic Finsler connections of hypersurfaces and Finslerien projective geometry, *J. Math. Kyoto Univ.,* **25** (1985), 107–144.

[17] M. Matsumoto : Foundations of Finsler geometry and special Finsler spaces, (1986), *Kaiseisha Press, Japan.*

[18] M. Matsumoto : Theory of Y-extremal hypersurface and minimal hypersurfaces in a Finsler space, *J. Math. Kyoto Univ.,* to appear.

[19] R. Miron : Metrical Finsler structures and metrical Finsler connections, *J. Math. Kyoto Univ.* **23** (1983), 219–224.

[20] R. Miron : A non-standard theory of hypersurfaces in Finsler spaces, *An, St. Univ. Al. I. Cuza. Iaşi, 1984.*

[21] R. Miron and A. Bejuncu : A new method in geometry of Finsler subspaces, *An. St. Univ. Al. I. Cuza Iaşi,* **30** (1984), 55–59.

[22] A. Rapcsák : Eine neue Charakterisierung Finslersche Räume skalare und Konstanter Krümmung und projectiv-ebene Räume, *Acta Math. Acad. Sci. Hunger.*, **8** (1957), 1–18.

[23] H. Rund : Über die Parallelverschiebung in Finslerschen Räumen, *Math. Z.* **54** (1951), 115–128.

[24] H. Rund : The theory of subspaces in a Finsler space Ⅰ, Ⅱ, *Math. Z.*, **56** (1952), 363–375 ; **57** (1953), 193–210.

[25] H. Rund : Hypersurfaces of a Finsler space, *Canad. J. Math.* **8** (1956), 487–503.

[26] H. Rund : The differential geometry of Finsler spaces, *Grundlehr. Math. Wiss.*, **101** (1959), *Springer—Verlag.*

[27] H. Rund : Curvature properties of hypersurfaces of Finsler and Minkowskian spaces, *Tensor, N. S.*, **14** (1963), 226–244.

[28] Z. I. Szabó : Ein Finslerschen Raum ist gerade dann von skalarer Krümmung, wenn seine Weylsche Projectivkrümmung verschwindet, *Acta Sci. Math. Szeged*, **39** (1977), 163–168.

[29] O. Varga : Über der inneren und induzierten Zusammenhang für Hyperflächen in Finslerschen Räume, *Publ. Math. Debrecen*, **8** (1961), 208–217.

[30] H. Yasuda : On TM-connections of a Finsler space and the induced TM-connections of its hypersurfaces, *Proc. Rom. -Jap. Col. Finsler Geo., Braşov*, (1984), 183–232.

[31] H. Yasuda : A study of connections in a Finsler space, *Ann. Rep. Asahikawa Med. Coll.*, **7** (1986), 1–17.

[32] H. Yasuda : On connections with deflexion tensors in a Finsler space, to appear.

[33] H. Yasuda and M. Yoshida : On special hypersurfaces of a Finsler space, *Tensor, N. S.*, **42** (1985), 283–298.

[34] H. Yasuda and M. Yoshida : A study of connections on hypersurfaces in a Finsler space, to appear.

[35] M. Yoshida : Curvature tensors on hypersurfaces of a Finsler space endowed with TM-connections, *Memoirs of Sagami Inst. Tech.*, **19** (1985), 65–83.

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