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ON *TM*-CONNECTIONS OF A FINSLER SPACE AND THE INDUCED *TM*-CONNECTIONS ON ITS HYPERSURFACES

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Introduction. In the previous paper [7][1], we introduced *TM*-connections on an n-dimentional Finsler space from the standpoint of tangent Minkowski spaces. Successively we developed the theory of these connections in papers ([**8**], [**9**], [**10**]). However, we have not yet the axiomatic theory of them. The first purpose of the present paper is to construct such a theory.

The theory of subspaces of a Finsler space has been investigated by many authors. Most of the induced connections treated by them are axiomatically different from the original connections. This makes not only the theory of subspaces more complicated ,but also its geometrical survey unclear. Such circumstances are geometrically undesirable. The second purpose of this paper is to get over the above circumstances to some extent. For this purpose we construct a theory of hypersurfaces by the use of *TM*-connections. One of the most important results is that the induced *TM*-connection is a *TM*-connection (in general not intrinsic) on a hypersurface. The terminologies and notations refer to the papers ([**7**] ~ [**10**]) unless otherwise stated.

§ 1. ***TM*-connections**. Let M_n be an n-dimentional Finsler space with a fundamental function $L(x,y)$ and be endowed with a *TM*-connection $TM\Gamma=(\Gamma^i_{j\,k}, \Gamma^i_{\ k}, C^i_{j\,k})$. This connection is defined as follows :

Firstly the v-connection is given by

$$(1.1)\qquad C^i_{j\,k} = g^{ih} C_{jhk}, \quad C_{jhk} = \tfrac{1}{2}\partial g_{jh} / \partial y^k,$$

1) Numbers in brackets refer to the references at the end of the paper.

where $g_{jh} = \frac{1}{2}\partial^2 L^2/\partial y^j \partial y^h$. Secondly the non-linear connection is given by

$$(1.2)\qquad \Gamma^i_{\ k} = y^j \Gamma^{\ i}_{j\ k} = G^i_{\ k} + T^i_{\ k},$$

where $G^i_{\ k}$ is the non-linear connection of Cartan (or Berwald) and

(a) $T^i_{\ k}$ is a positively homogeneous tensor of degree 1 in y^i such that $y_i T^i_{\ k} = 0$.

Thirdly the *h*-connection is given by

$$(1.3)\qquad \Gamma^{\ i}_{j\ k} = \partial \Gamma^i_{\ k}/\partial y^j + Q^{\ i}_{j\ k} = G^{\ i}_{j\ k} + T^{\ i}_{j\ k} + Q^{\ i}_{j\ k},$$

where $G^{\ i}_{j\ k}$ is the *h*-connection of Berwald, $T^{\ i}_{j\ k} = \partial T^i_{\ k}/\partial y^j$ and

(b) $Q^{\ i}_{j\ k}$ is a positively homogeneous tensor of degree 0 in y^i such that $y_i Q^{\ i}_{j\ k} = y^j Q^{\ i}_{j\ k} = 0$.

The absolute differentials of vectors y^i and $X^i(x,y)$ on M_n are difined as follows:

$$(1.4)\qquad Dy^i = dy^i + \Gamma^i_{\ k}\, dx^k,$$

$$(1.5)\qquad \begin{aligned} DX^i &= dx^i + (\Gamma^{\ i}_{j\ k} + C^{\ i}_{j\ h}\Gamma^h_{\ k})X^j dx^k + C^{\ i}_{j\ k} X^j dy^k \\ &= X^i_{\ |k}\, dx^k + X^i|_k\, Dy^k, \text{ where} \end{aligned}$$

$$(1.5)_1\qquad X^i_{\ |k} = \partial X^i/\partial x^k - \Gamma^j_{\ k}\partial X^i/\partial y^j + \Gamma^{\ i}_{j\ k} X^j,$$

$$(1.5)_2\qquad X^i|_k = \partial X^i/\partial y^k + C^{\ i}_{j\ k} X^j.$$

Now we shall consider axioms characterizing this ***TMΓ***. First we can state

Theorem 1.1. *A TM-connection Γ is characterized by the following six axioms* **(TM1) ~ (TM6)**:

(TM1) *The Γ is metrical,* i.e., $L_{|k} = \partial L/\partial x^k - \Gamma^j_{\ k}\partial L/\partial y^j = 0$.

(TM2) *The deflection tensor* $D^i_{\ k}$ *vanishes, namely* $y^j \Gamma^{\ i}_{j\ k} - \Gamma^i_{\ k} = 0$.

(TM3) *The Γ is v-metrical,* i.e., $g_{ij}|_k = \partial g_{ij}/\partial y^k - C^{\ r}_{i\ k} g_{rj} - C^{\ r}_{j\ k} g_{ir} = 0$.

(TM4) *The v-torsion tensor vanishes, namely* $C^{\ i}_{j\ k} - C^{\ i}_{k\ j} = 0$.

(TM5) *The absolute differential of* y_i $(= g_{ij} y^j)$ *is given by* $Dy_i = g_{ij} Dy^j$.

(TM6) *Paths with respect to the Γ are always geodesics of* M_n

Proof. If a connection Γ is a *TM*-connection, then it follows from (1.1) ~ (1.3) that the Γ satisfies all the axioms **(TM1)** ~ **(TM6)**. Conversely suppose that a connection $\Gamma = (\Gamma^{\ i}_{j\ k}, \Gamma^i_{\ k}, \tilde{C}^{\ i}_{j\ k})$ on M_n satisfies **(TM1)**~**(TM6)**. Then we first have $\tilde{C}^{\ i}_{j\ k} = C^{\ i}_{j\ k}$ (in (1.1)) because of **(TM3)** and **(TM4)**. Since this Γ has the *hv*-torsion tensor $\tilde{P}^{\ i}_{j\ k}$, a tensor $Q^{\ i}_{j\ k}$ is uniquely determined as follows:

(1. 6) $\Gamma^{i}_{j\,k} = \partial \Gamma^{\,i}_{\ k} / \partial y^{j} + Q_{j\,k}^{\,i}$,

provided that $Q_{j\,k}^{\,i} (= -\widetilde{P}_{j\,k}^{\,i})$ is a positively homogeneous tensor of degree 0 in y^{i}. Then contracting (1. 6) by y^{j} and making use of **(TM2)**, we have $y^{j} Q_{j\,k}^{\,i} = 0$. From **(TM1)** we obtain $(\Gamma^{i}_{\ k} - G^{i}_{\ k}) y_{i} = 0$. Therefore there exists a unique tensor $T^{i}_{\ k}$ such that $T^{i}_{\ k} y_{i} = 0$ and

(1. 7) $\Gamma^{i}_{\ k} = G^{i}_{\ k} + T^{i}_{\ k}$.

Contracting (1. 7) by y^{k}, we have $\Gamma^{i}_{\ k} y^{k} = 2G^{i} + T^{i}_{\ k} y^{k}$. In this case we have $\Gamma^{i}_{\ k} y^{k} = 2G^{i}$ because of **(TM6)**. Therefore we obtain $T^{i}_{\ k} y^{k} = 0$. Further if we differentiate (1. 7) by y^{j} and substitute the result in (1. 6), then we have

(1. 8) $\Gamma^{i}_{j\,k} = G_{j\,k}^{\,i} + T_{j\,k}^{\,i} + Q_{j\,k}^{\,i}$.

Since $Dy_{i} = y^{l} Dg_{ij} + g_{ij} Dy^{j}$, the axiom **(TM5)** is equivalent to **(TM5)′**: *The absolute differential of* g_{ij} *is indicatric,* i.e., $y^{j} Dg_{ij} = 0$.

On the other hand, it follows from **(TM3)** that $Dg_{ij} = g_{ij|k} dx^{k}$. Therefore if we apply this result to **(TM5)′**, then we obtain

$$(1.9)\qquad \begin{aligned} &(\partial g_{ij} / \partial x^{k} - \Gamma^{h}_{\ k} \partial g_{ij} / \partial y^{h} - g_{rj} \Gamma^{\,r}_{i\,k} - g_{ir} \Gamma^{\,r}_{j\,k}) y^{j} \\ &= y_{r} G_{i\,k}^{\,r} + g_{ir} G^{r}_{\ k} - y_{r} \Gamma^{\,r}_{i\,k} - g_{ir} \Gamma^{r}_{\ k} = 0. \end{aligned}$$

Applying (1. 7) and (1. 8) to (1. 9) and noticing $y_{r} T_{i\,k}^{\,r} = -g_{ir} T^{r}_{\ k}$, we have $y_{i} Q_{j\,k}^{\,i} = 0$. Thus the connection Γ in consideration becomes a *TM*-connection. Q. E. D.

Immediately we can state

Corollary 1. 1. 1. *Given tensors* $T^{i}_{\ k}$ *and* $Q_{j\,k}^{\,i}$ *satisfying* (a) *and* (b) *respectively, a TM-connection is uniquely determined by the axioms* **(TM1)**～**(TM6)**.

Corollary 1. 1. 2. *The Hashiguchi connection* $H\Gamma$ *is a special TM-connection determined by* $T^{i}_{\ k} = 0$ *and* $Q_{j\,k}^{\,i} = 0$, *and it is also characterized by the following six axioms* ([2] ～[4]):

(TM1), **(TM2)**, **(TM3)**, **(TM4)**, **(STM6)** (in § 2), **(GT5)** (in § 2).

Corollary 1. 1. 3. *The Cartan connection* $C\Gamma$ *is a special TM-connection* *determined by* $T^i_{\ k}=0$ *and* $Q_j{}^i{}_k=-P_j{}^i{}_k$, *and it is also characterized by the following five axioms* ([2], [3]): **(TM2)**, **(TM3)**, **(TM4)**, {**(STM6)**, **(RTM1)**} (in §2).

We shall call a *TM*-connection a *WTM-connection* if the tensor $T^i_{\ k}$ in (1. 2) is defined by the following weak condition:
(c) $T^i_{\ k}$ is a positively homogeneous tensor of degree 1 in y^i such that $T^i_{\ k}y_i=0$.

Then we can state

Theorem 1. 2. *A WTM-connection is characterized by the following five axioms:* **(TM1)**, **(TM2)**, **(TM3)**, **(TM4)**, **(TM5)**.

Given tensors $T^i_{\ k}$ *and* $Q_j{}^i{}_k$ *satisfying* (c) *and* (b) *respectively, a WTM-connection is uniquely determined by the above five axioms.*

Proof. The axiom **(TM6)** implies $T^i_{\ k}y^k=0$ alone. Therefore the proof of Theorem 1.1 except this fact is still valid for that of the present theorem.

§2. **Special TM-connections**. In this section, we shall treat special *TM*-connections. First we consider an *r*-metrical *TM*-connection, namely a *TM*-connection satisfying $Dg_{ij}=0$. For this, it is necessary and sufficient that the following equation holds ([7] ~[9]):

$$T_{ijk}+T_{jik}+Q_{ijk}+Q_{jik}+2(C_{ijr}T^r_{\ k}+P_{ijk})=0, \tag{2. 1}$$

where $T_{ijk}=g_{jr}T_i{}^r{}_k$ and $Q_{ijk}=g_{jr}Q_i{}^r{}_k$.

We shall call a *TM*-connection Γ an *RTM-connection* if the Γ is *r*-metrical, that is, an equation (2. 1) holds for the Γ. Then we can state

Theorem 2. 1. *An RTM-connection* Γ *is characterized by the following five axioms* **(RTM1)** ~**(RTM5)**:

(RTM1) *The* Γ *is h-metrical, namely* $g_{ij|k}=0$.

(RTM2) =**(TM2)**, **(RTM3)**=**(TM3)**, **(RTM4)**=**(TM4)**, **(RTM5)**=**(TM6)**.

Proof. If a connection Γ is an *RTM*-connection, then it is easily verified from (1. 1)~(1. 3) and (2. 1) that the Γ satisifies **(RTM1)**~**(RTM5)**. Conversely suppose that a connection $\Gamma=(\Gamma_j{}^i{}_k, \Gamma^i_{\ k}, \tilde{C}_j{}^i{}_k)$ satisfies **(RTM 1)**~**(RTM5)**. Then in just the same way as in the proof of Theorem 1.1, we have $\tilde{C}_j{}^i{}_k=C_j{}^i{}_k$ and (1. 6) together with $Q_j{}^i{}_k y^j=0$. Therefore from

(**RTM1**) and (**RTM3**) we obtain $Dg_{ij}=0$ and hence (**TM5**)′ holds. By virtue of (**RTM2**) we have $y^i{}_{|k}=0$, from which and (**RTM1**) it follows that $g_{ij|k}y^iy^j=L^2{}_{|k}=2LL_{|k}=0$, that is, (**TM1**) holds. If we further take account of (**RTM5**), then we can conclude that the connection Γ in consideration is an r-metrical TM-connection, namely an RTM-connection. In this case, the equation (2. 1) holds. Q. E. D.

Now if we put

$$Z_{ijk}=Q_{ijk}-Q_{jik}, \tag{2. 2}$$

then this tensor satisfies the following condition:

(d) Z_{ijk} is a positively homogeneous tensor of degree 0 in y^i such that $Z_{ijk}y^i=Z_{ijk}y^j=0$ and $Z_{ijk}+Z_{jik}=0$.

Then from (2. 1) and (2. 2) we have

$$Q_j{}^i{}_k=\tfrac{1}{2}g^{ir}(Z_{jrk}-T_{jrk}-T_{rjk})-C_j{}^i{}_rT^r{}_k-P_j{}^i{}_k, \tag{2. 3}$$

substitution of which in (1. 3) yields

$$\Gamma_j{}^i{}_k=\Gamma^*{}_j{}^i{}_k-C_j{}^i{}_rT^r{}_k+\tfrac{1}{2}g^{ir}(Z_{jrk}+T_{jrk}-T_{rjk}), \tag{2. 4}$$

where $\Gamma^*{}_j{}^i{}_k$ is the h-connection of Cartan. Hence we can state

Corollary 2. 1. 1. *Given tensors* $T^i{}_k$ *and* Z_{ijk} *satisfying* (a) *and* (d) *respectively, an RTM-connection is uniquely determined by the axioms* (**RTM1**)～(**RTM5**). *In this case, the h-connection is given by* (2. 4).

Next let the h-torsion tensor $\tau_j{}^i{}_k$ (or τ_{jik}) be given, namely

$$\begin{aligned}\tau_j{}^i{}_k&=T_j{}^i{}_k+Q_j{}^i{}_k-(T_k{}^i{}_j+Q_k{}^i{}_j),\\ \tau_{jik}&=T_{jik}+Q_{jik}-(T_{kij}+Q_{kij}).\end{aligned} \tag{2. 5}$$

Then from (2. 1) and (2. 5) we obtain

$$\tau_{jik}+\tau_{ijk}+\tau_{ikj}=2(T_{ikj}+Q_{ikj}+C_{ikr}T^r{}_j+C_{jkr}T^r{}_i-C_{ijr}T^r{}_k+P_{ijk}), \tag{2. 6}$$

contraction of which by g^{hk} yields

$$T_i{}^h{}_j+Q_i{}^h{}_j+Q_i{}^h{}_j=\tfrac{1}{2}g^{hk}(\tau_{jik}+\tau_{ijk}+\tau_{ikj})-C_i{}^h{}_rT^r{}_j-C_j{}^h{}_rT^r{}_i \tag{2. 7}$$

$$+ g^{hk} C_{ijr} T^r_{\ k} - P_{i\ j}^{\ h}.$$

Contracting (2. 7) by y^i, we have

$$T^h_{\ j} = \tfrac{1}{2} g^{hk} (\tau_{j0k} + \tau_{0jk} + \tau_{0kj}). \tag{2. 8}$$

Applying (2. 7) to (1. 3), we obtain

$$\begin{aligned} \Gamma_{j\ k}^{\ i} = \Gamma^{*\ i}_{j\ k} + \tfrac{1}{2} g^{ir} (\tau_{jkr} + \tau_{jrk} + \tau_{kjr}) + g^{ih} C_{jkr} T^r_{\ h} - C_{j\ r}^{\ i} T^r_{\ k} \\ - C_{k\ r}^{\ i} T^r_{\ j}. \end{aligned} \tag{2. 9}$$

Contracting (2. 8) by y^j, we obtain $\tau_{00k} = \tau_{0\ k}^{\ 0} = 0$.

Consequently we can state

Corollary 2. 1. 2. *Given the h-torsion tensor* $\tau_{j\ k}^{\ i}$ *satisfying* $\tau_{0\ k}^{\ 0} = 0$, *an RTM-connection is uniquely determined by the axioms* **(RTM1)**~**(RTM5)**. *In this case, the h-connection is given by* (2. 9), *provided that* $T^i_{\ k}$ *is given by* (2. 8).

An *r*-metrical *WTM*-connection is called an *RWTM-connection* or a *generalized Cartan τ-connection* [3]. We shall denote this connection by $C\Gamma(\tau)$. Then this $C\Gamma(\tau)$ is characterized by the following four axioms **(CT1)**~**(CT4)** : **(CT1)**=**(RTM1)**, **(CT2)**=**(RTM2)**, **(CT3)**=**(RTM3)**, **(CT4)**=**(RTM4)**.

With respect to $C\Gamma(\tau)$ the contracted vector $T^i_{\ 0}$ does, in general, not vanish. Therefore contracting (2. 7) by y^i, we obtain

$$T^h_{\ j} = \tfrac{1}{2} g^{hk} (\tau_{j0k} + \tau_{0jk} + \tau_{0kj}) - C_{j\ r}^{\ h} T^r_{\ 0}, \quad T^h_{\ 0} = g^{hk} \tau_{00k}. \tag{2. 10}$$

Consequently we can state

Theorem 2. 2 (Hashiguchi, 1975). *An RWTM-connection (or a* $C\Gamma(\tau)$*) is characterized by the axioms* **(CT1)**~**(CT4)**. *Given the h-torsion tensor* $\tau_{j\ k}^{\ i}$, *an RWTM-connection (or* $C\Gamma(\tau)$*) is uniquely determined by* (CT1)~(CT4). In *this case, the h-connection is given by* (2. 9), *provided that* $T^i_{\ k}$ *is given by* (2. 10).

Corresponding to Corollary 2.1.1, we have

Corollary 2. 2. 1. *Given tensors* $T^i_{\ k}$ *and* Z_{ijk} *satisfying* (c) *and* (d) *respectively, an RWTM-connection (or a* $C\Gamma(\tau)$*) is uniquely determined by* **(CT1)**~**(CT4)**. *In this case, the h-connection is given by* (2. 4).

Note 2. 1. The Cartan connection $C\Gamma$ is also a special RTM-connection or a special $RWTM$-connection defined as follows respectively:

(1) $T^i{}_k=0$ and $Z_{ijk}=0$ or (2) $\tau_j{}^i{}_k=0$.

Let us consider a symmetric TM-connection. We shall call a TM-connection an *STM-connection* if the h-torsion tensor $\tau_j{}^i{}_k$ vanishes.

Then we can state

Theorem 2. 3. *An STM-connection is characterized by the following six axioms* **(STM1)**~**(STM6)**: **(STMi)** = **(TMi)** (**i**=1, 2, . . . , 5),

(STM6) *The h-torsion tensor* $\tau_j{}^i{}_k$ *vanishes, namely* $\Gamma_j{}^i{}_k=\Gamma_k{}^i{}_j$.

Proof. From **(STM1)**~**(STM5)** we have

(2. 11) $\qquad \Gamma^i{}_k=G^i{}_k+T^i{}_k,\ \Gamma_j{}^i{}_k=G_j{}^i{}_k+T_j{}^i{}_k+Q_j{}^i{}_k,$

provided that $T^i{}_k$ satisfies the condition (c), while $Q_j{}^i{}_k$ satisfies (b).

By virtue of **(STM6)** and (2.11) we have

(2. 12) $\qquad T_j{}^i{}_k+Q_j{}^i{}_k=T_k{}^i{}_j+Q_k{}^i{}_j,$

contraction of which by $y_i\,y^j$ yields $T_{k0}=0$. Therefore the axiom **(TM6)** is satisfied. Q. E. D.

If we now put

(2. 13) $\qquad U_j{}^i{}_k=Q_j{}^i{}_k+Q_k{}^i{}_j,$

then from (2. 12) and (2. 13) we obtain

(2. 14) $\qquad Q_j{}^i{}_k=\tfrac{1}{2}(U_j{}^i{}_k+T_k{}^i{}_j-T_j{}^i{}_k).$

Substituting (2. 14) in the second expression of (2. 11), we have

(2. 15) $\qquad \Gamma_j{}^i{}_k=G_j{}^i{}_k+\tfrac{1}{2}(U_j{}^i{}_k+T_j{}^i{}_k+T_k{}^i{}_j),$

contraction of which by y^j, because of (2. 13), yields

(2. 16) $\qquad \Gamma^i{}_k=G^i{}_k+T^i{}_k,\ T^i{}_k=\tfrac{1}{2}Q_k{}^i{}_0.$

Given a tensor $Q_j{}^i{}_k$ satisfying (b), from (2.16) we obtain $T^i{}_k(=\tfrac{1}{2}Q_k{}^i{}_0)$ and hence $T_j{}^i{}_k(=\partial T^i{}_k/\partial y^j)$. If we denote $\partial Q_k{}^i{}_0/\partial y^j$ by $Q_k{}^i{}_{0\|j}$, then from (2. 13) and (2. 15) we obtain

(2. 17) $\Gamma_{j\ k}^{\ i} = G_{j\ k}^{\ i} + \frac{1}{2}(Q_{j\ k}^{\ i} + Q_{k\ j}^{\ i}) + \frac{1}{4}(Q_{k\ 0\|j}^{\ i} + Q_{j\ 0\|k}^{\ i})$

Consequently we can state

Corollary 2. 3. 1. *Given a tensor* $Q_{j\ k}^{\ i}$ *satisfying* (b), *an STM-connection is uniquely determined by the five axioms* (**STM1**)~(**STM4**) *and* (**STM6**). In *this case, the h-connection is given by* (2. 17).

We shall consider another determination of $\Gamma_{j\ k}^{\ i}$. Contracting (2. 12) by y^j and differentiating the result by y^j, we have

(2. 18) $T^i_{\ k} = \frac{1}{2} Q_{k\ 0}^{\ i}$, $T_{j\ k}^{\ i} = \frac{1}{2} Q_{k\ 0\|j}^{\ i}$,

which, because of (2. 11), implies

(2. 19) $\Gamma_{j\ k}^{\ i} = G_{j\ k}^{\ i} + \frac{1}{2} Q_{k\ 0\|j}^{\ i} + Q_{j\ k}^{\ i}$.

Since $\Gamma_{j\ k}^{\ i} = \Gamma_{k\ j}^{\ i}$, the following equation holds:

(2. 20) $y^r (Q_{j\ r\|k}^{\ i} - Q_{k\ r\|j}^{\ i}) = 0.$

Hence we have

Corollary 2. 3. 2. *Given a tensor* $Q_{j\ k}^{\ i}$ *satisfying* (b) *and* (2. 20), *an ST M-connection is uniquely determined by the five axioms* (**STM1**)~ (**STM4**) *and* (**STM6**).

Note 2. 2. The Hashiguchi and Cartan connections are special STM-connections defined by $Q_{j\ k}^{\ i} = 0$ and $Q_{j\ k}^{\ i} = -P_{j\ k}^{\ i}$ respectively.

A *TM*-connection is called a *GT-connection* if the *hv*-torsion tensor vanishes. Then from (1. 2) and (1. 3) we have

(2. 21) $\Gamma_{j\ k}^{\ i} = G_{j\ k}^{\ i} + T_{j\ k}^{\ i}$, $y^j \Gamma_{j\ k}^{\ i} = \Gamma^i_{\ k} = G^i_{\ k} + T^i_{\ k}$.

Then we can state

Theorem 2. 4. *A GT-connection is characterized by the following five axioms* (**GT1**)~(**GT5**):(**GT1**)=(**TM1**), (**GT2**)=(**TM3**), (**GT3**) = (**TM4**), (**GT4**) = (**TM6**),

(GT5) *The hv-torsion tensor vanishes*, i.e., $\Gamma_{j\ k}^{\ i} - \Gamma^i_{\ k\|j} = 0$.

Given a tensor $T^i_{\ k}$ *satisfying* (a), *a GT-connection is uniquely determined by the axioms* (**GT1**)~(**GT5**).

Proof. The axiom (**GT5**) implies (**TM2**). From (**GT1**) and (**GT5**) we have (2. 21) with $T^{0}{}_{k}=0$. Therefore it follows from (**GT1**), (**GT3**) and $Q_{j}{}^{i}{}_{k}=0$ that the axiom (**TM5**)′ holds. Lastly from (**GT4**) we have $T^{i}{}_{0}=0$.

Now let the h-torsion tensor $\tau_{j}{}^{i}{}_{k}$ be given, namely

$$(2.\ 22)\qquad \tau_{j}{}^{i}{}_{k}=T_{j}{}^{i}{}_{k}-T_{k}{}^{i}{}_{j}.$$

Contracting (2. 22) by y^{j}, we have

$$(2.\ 23)\qquad T^{i}{}_{k}=\tfrac{1}{2}\tau_{0}{}^{i}{}_{k},$$

contraction of which by y_{i} yields $\tau_{0}{}^{0}{}_{k}=0$. From (2. 21) and (2. 23) we have

$$(2.\ 24)\qquad \Gamma_{j}{}^{i}{}_{k}=G_{j}{}^{i}{}_{k}+\tfrac{1}{2}\tau_{0}{}^{i}{}_{k\|j},\ \Gamma^{i}{}_{k}=G^{i}{}_{k}+\tfrac{1}{2}\tau_{0}{}^{i}{}_{k}.$$

Consequently we have

Corollary 2. 4. 1. *Given the h-torsion tensor* $\tau_{j}{}^{i}{}_{k}$ *satisfying* $\tau_{0}{}^{0}{}_{k}=0$, *a GT-connection is uniquely determined by the five axioms* (**GT1**)~(**GT5**). *In this case, the h-connection and non-linear connection are given by* (2. 24).

We shall call a *GT*-connection a *WGT-connection* if the tensor $T^{i}{}_{k}$ in (2. 21) is defined by the condition (c). Then we can state

Theorem 2. 5. *A WGT-connection is characterized by the following four axioms*: (**GT1**), (**GT2**), (**GT3**), (**GT5**).

Given a tensor $T^{i}{}_{k}$ *satisfying* (c), *a WGT-connection is uniquely determined by the above four axioms.*

Contracting (2. 22) by y^{j}, because of $T^{i}{}_{k}=T^{i}{}_{0\|k}-T_{k}{}^{i}{}_{0}$ we have

$$(2.\ 25)\qquad T^{i}{}_{k}=\tfrac{1}{2}(\tau_{0}{}^{i}{}_{k}+T^{i}{}_{0\|k}).$$

Further if we contract (2 . 25) by y_{i}, then we have $\tau_{0}{}^{0}{}_{k}=T_{k0}$. Therefore applying this result to (2. 25), we obtain

$$(2.\ 26)\qquad T^{i}{}_{k}=\tfrac{1}{2}(\tau_{0}{}^{i}{}_{k}-\tau^{i}{}_{00\|k}),\ \tau^{i}{}_{00}=g^{ir}\tau_{r00}.$$

Then from (2. 21) and (2. 26) we have

$$(2.\ 27)\qquad \Gamma_{j}{}^{i}{}_{k}=G_{j}{}^{i}{}_{k}+\tfrac{1}{2}(\tau_{0}{}^{i}{}_{k\|j}-\tau^{i}{}_{00\|k\|j}),\ \Gamma^{i}{}_{k}=G^{i}{}_{k}+\tfrac{1}{2}(\tau_{0}{}^{i}{}_{k}-\tau^{i}{}_{00\|k}).$$

Consequently we can state

Corollary 2. 5. 1. *Given the h-tosion tensor* $\tau_{j}{}^{i}{}_{k}$, *a WGT-connection is* *uniquely determined by the four axioms* **(GT1)**, **(GT2)**, **(GT3)** *and* **(GT5)**. *In this case, the h-connection and non-linear connection are given by* (2. 27).

Note 2.3. The Hashiguchi connection is a special *GT*(or *WGT*)-connection defined by $T^i{}_k=0$ (or $\tau_j{}^i{}_k=0$).

An *r*-metrical *GT*-connection is called the *IS-connection.* For this connection, it follows from (2. 1) that the following equation holds:

$$T_{ijk}+T_{jik}+2(C_{ijr}T^r{}_k+P_{ijk})=0. \tag{2. 28}$$

Then we can state

Theorem 2. 6. *The IS-connection is characterized by the following five axioms* **(IS1)**$\sim$**(IS5)**:**(ISi)**=**(GTi)**(i=**1, 2, 3, 4**),

(IS5) *The hv-curvature tensor* $\widetilde{P}_j{}^i{}_{kh}$ *vanishes, namely*

$$\widetilde{P}_j{}^i{}_{kh}=\Gamma_j{}^i{}_{k\|h}-C_j{}^i{}_{h|k}+C_j{}^i{}_r\widetilde{P}^r{}_{hk}=0,\ \widetilde{P}^r{}_{hk}=\Gamma^r{}_{k\|h}-\Gamma_h{}^r{}_k.$$

Proof. It is known [8] that with respect to a *TM*-connection the *hv*-curvature tensor $\widetilde{P}_j{}^i{}_{kh}$ vanishes if and only if $\widetilde{P}_j{}^i{}_k(=-Q_j{}^i{}_k)=0$ and equation (2. 28) holds. Therefore if a connection Γ is the IS-connection, then it is easily verified that the Γ satisfies **(IS1)**$\sim$**(IS5)**. Conversely suppose that a connection $\Gamma=(\Gamma_j{}^i{}_k,\ \Gamma^i{}_k,\ \widetilde{C}_j{}^i{}_k)$ satisfies **(IS1)**$\sim$**(IS5)**. Then from **(IS2)** and **(IS3)** we first have $\widetilde{C}_j{}^i{}_k=C_j{}^i{}_k$. Next, by virtue of **(IS1)**, **(IS4)** and **(IS5)** we obtain (2. 21) and (2. 28) together with $T^0{}_k=T^i{}_0=0$.

Hence the Γ becomes the *IS*-connection. Q. E. D.

We shall call an r-metrical *WGT*-connection the *RWGT-connection.* Then we have

Corollary 2. 6. 1. *The RWGT-connection is characterized by the following four axioms*:**(RTM1)**, **(TM3)**, **(TM4)**, **(GT5)**. *Then the h-connection and non-linear connection are given by* (2. 21) *together with* (2. 28).

Under the *RWGT*(or *IS*)-connection, indicatrices considered as Riemannian spaces are isometric.

Now we shall consider a special *RWTM*-connection which is called a Wagner connection [2], [5]. This connection is characterized by the following five axioms **(W1)**$\sim$**(W5)**:**(Wi)**=**(CTi)**(i=1, 2, 3, 4),

(W5) The *h*-torsion tensor $\tau_j{}^i{}_k$ is given by

(2. 29) $\qquad \tau_j{}^i{}_k = \delta^i{}_j s_k - \delta^i{}_k s_j,$

where s_k is a positively homogeneous covariant vector of degree 0 in y^i.

According to Theorem 2.2 we can deduce

(2. 30) $\qquad T^i{}_k = s^i y_k - \delta^i{}_k s_0 - L^2 C_k{}^i{}_r s^r,$

(2. 31) $\qquad \Gamma_j{}^i{}_k = \Gamma^*{}_j{}^i{}_k + g_{jk} s^i - \delta^i{}_k s_j + C_j{}^i{}_k s_0 + (y^i C_{jkr} s^r - C_j{}^i{}_r s^r y_k - C_k{}^i{}_r s^r y_j) + L^2 (C_j{}^i{}_r C_k{}^r{}_t + C_k{}^i{}_r C_j{}^r{}_t - C_r{}^i{}_t C_j{}^r{}_k) s^t,$

(2. 32) $\qquad \Gamma^i{}_k = y^j \Gamma_j{}^i{}_k = G^i{}_k + s^i y_k - \delta^i{}_k s_0 - L^2 C_k{}^i{}_r s^r.$

Now we assume that $\tau_j{}^0{}_k = y_i \tau_j{}^i{}_k = 0$. Then from (2. 29) we have $y_j s_k - y_k s_j = 0$. Therefore there exists a positively homogeneous scalar $f(x, y)$ of degree 0 in y^i such that

(2. 33) $\qquad s_j = -f y_j / L \;\; or \;\; s^j = -f y^j / L.$

Then (2. 30) ~ (2. 32) are reducible to

(2. 34) $\qquad \Gamma^i{}_k = G^i{}_k + T^i{}_k, \;\; T^i{}_k = f L h^i{}_k,$

(2. 35) $\qquad \Gamma_j{}^i{}_k = \Gamma^*{}_j{}^i{}_k + f(l_j h^i{}_k - l^i h_{jk} - L C_j{}^i{}_k).$

The above expressions show that the connection in consideration becomes an *AMR*-connection [9]. Conversely we assume that a Wagner connection is an *AMR*-connection. Then from (2. 34) we have

(2. 36) $\qquad \tau_j{}^i{}_k = f(l_j \delta^i{}_k - l_k \delta^i{}_j),$

contraction of which by y_i yields $\tau_j{}^0{}_k = 0$. Consequently we can state

Theorem 2. 7. *A Wagner connection is an AMR-connection if and only if the h-torsion tensor* $\tau_j{}^i{}_k$ *is indicatric with respect to the upper index i,* i. e., $\tau_j{}^0{}_k = 0$.

Further we can state

Corollary 2. 7. 1. *An AMR-connection is a special Wagner connection such that the h-torsion tensor* $\tau_j{}^i{}_k$ *is indicatric with respect to i.*

Corollary 2. 7. 2. *An AMR-connection is uniquely determined by the following five axioms* (**AMR1**) ~ (**AMR5**) : (**AMRi**) = (**Wi**) (i = 1, 2, 3, 4).

(**AMR5**) *The h-torsion tensor* $\tau_j{}^i{}_k$ *is given by* (2. 36), *provided* f *is a positively homogeneous scalar of degree* 0 *in* y^i .

§3. ***TM*-connections without *v*-connection.** In this section, we shall treat a *TM*-connection $\Gamma=(\Gamma_j{}^i{}_k,\ \Gamma^i{}_k,\ 0)$ without v-connection and denote it by $TM\Gamma(0)$. In this case, the non-linear connection $\Gamma^i{}_k$ and h-connection $\Gamma_j{}^i{}_k$ are still defined by (1. 2) and (1. 3) respectively. The absolute differentials of vectors y^i and $X^i(x, y)$ are defined as follows:

$$Dy^i = dy^i + \Gamma^i{}_k dx^k, \tag{3. 1}$$

$$\overset{0}{D}X^i = dX^i + \Gamma_j{}^i{}_k X^j dx^k = X^i{}_{|k} dx^k + X^i{}_{\|k} Dy^k, \tag{3. 2}$$

$$X^i{}_{|k} = \partial X^i / \partial x^k - \Gamma^j{}_k X^i{}_{\|j} + \Gamma_j{}^i{}_k X^j,\quad X^i{}_{\|k} = \partial X^i / \partial y^k. \tag{3. 3}$$

We shall call a *TM*-connection $\Gamma=(\Gamma_j{}^i{}_k,\ \Gamma^i{}_k,\ 0)$ a *TM*(0)*-connection.* Then we can state

Theorem 3. 1. *A TM*(0)*-connection* Γ *is characterized by the following five axioms* (**T1**)～(**T5**):

(**T1**)＝(**TM1**), (**T2**)＝(**T3**)＝(**TM5**)(*or* (**TM5**)′), (**T4**)＝(**TM6**),

(**T5**) *The v-connection parameters vanish*, i. e., $C_j{}^i{}_k=0$.

Given tensors $T^i{}_k$ *and* $Q_j{}^i{}_k$ *satisfying* (a) *and* (b) *respectively, a TM*(0)*-connection is uniquely determined by* (**T1**)～(**T5**).

Proof. From (3. 1)～(3. 3) we have

$$\overset{0}{D}g_{ij} = g_{ij|k} dx^k + 2C_{ijk} Dy^k, \tag{3. 4}$$

which implies (1. 9). Therefore if we neglect the v-connection, then the remaining proof is just the same as in case of Theorem 1. Q. E. D.

We shall call a *TM*(0)-connection a *WTM*(0)-connection if the tensor $T^i{}_k$ in (1. 2) is defined by the condition (c). Then we have

Corollary 3. 1. 1. *A WTM*(0)*-connection is characterized by the following four axioms:* (**T1**), (**T2**), (**T3**), (**T5**). *Given tensors* $T^i{}_k$ *and* $Q_j{}^i{}_k$ *satisfying* (c) *and* (b) *respectively, a WTM*(0)*-connection is uniquely determined by the above four axioms.*

Corollary 3. 1. 2. *The Berwald connection is a special TM*(0)*-(or WTM*(0))*-connection defined by* $T^i{}_k=0$ *and* $Q_j{}^i{}_k=0$, *and it is also characterized by* the following four axioms: (**T1**), (**STM6**), (**GT5**), (**T5**).

Corollary 3. 1. 3. *The Rund connection is a special* $TM(0)$ *(or* $WTM(0)$*) -connection defined by* $T^i{}_k = 0$ *and* $Q_j{}^i{}_k = -P_j{}^i{}_k$, *and it is also characterized by the following four axioms*: (**RTM1**), (**T2**), (**STM6**), (**T5**).

We shall call a $TM(0)$-connection an $STM(0)$*-connection* if the h-connection is symmetric. Then corresponding to Theorem 2. 3, Corollary 2. 3. 1 and Corollary 2. 3. 2, we have

Theorem 3. 2. *An* $STM(0)$ *-connection is characterized by the following five axioms*: (**T1**), (**T2**), (**T3**), (**T5**), (**STM6**).

An $STM(0)$ *-connection is uniquely determined by the four axioms* (**T1**), (**T2**), (**T5**) *and* (**STM6**) *if a tensor* $Q_j{}^i{}_k$ *is given such that*

(1) $Q_j{}^i{}_k$ *satisfies* (b) *or* (2) $Q_j{}^i{}_k$ *satisfies* (b) *and* (2. 20).

Note 3. 1. An $STM(0)$- connection characterized by the above four axioms is also called a generalized Belwald P^1-connection [3].

Note 3. 2. The Berwald and Rund connections are also special $STM(0)$-connections defined by $Q_j{}^i{}_k = 0$ and $Q_j{}^i{}_k = -P_j{}^i{}_k$ respectively.

We shall call a $TM(0)$-connection an $RTM(0)$*-connection* if it is h-metrical. Then from Theorem 2. 1, Corollary 2. 1. 1 and Corollary 2. 1. 2, we can state

Theorem 3. 3. *An* $RTM(0)$ *-connection is characterized by the following four axioms*: (**RTM1**), (**RTM2**), (**RTM5**), (**T5**).

An $RTM(0)$ *-connection is uniquely determined by the above four axioms in the following case* ((1) *or* (2)):

(1) *Given tensors* $T^i{}_k$ *and* Z_{ijk} *satisfying* (a) *and* (d) *respectively.*

(2) *Given the h-torsion tensor* $\tau_j{}^i{}_k$ *satisfying* $\tau_0{}^0{}_k = 0$.

Note 3. 3. The Rund connection is a special $RTM(0)$-connection defined by $T^i{}_k = 0$ and $Z_{ijk} = 0$ (or $\tau_j{}^i{}_k = 0$).

Note 3. 4. Corresponding to Theorems 2. 2, 2. 4, 2. 5 and their corollaries, we can define many special $TM(0)$ (or $WTM(0)$)-connections. If we replace (**TM3**) and (**TM4**) with (**T5**), then most of the corresponding results in §2 are still valid for their connections.

§4. Hypersurfaces of $\mathbf{M}_n$**.** Let M_{n-1} be a hypersurface of M_n represented parametrically by the equation

(4. 1) $x^i = x^i (u^\alpha)$ $(i=1,2,\ldots,n;\ \alpha=1,2,\ldots,n-1)$,

where we suppose that the variables u^α form a coordinate system of M_{n-1}. In the following, Latin indices run from 1 to n, while Greek indices from 1 to $n-1$. We put

(4. 2) $B^i_\alpha = \partial x^i / \partial u^\alpha$,

and assume that the matrix (B^i_α) is of rank $n-1$. If we denote the components of a tangent vector y^i to a curve C contained in M_{n-1} by y^α in terms of u^α-system, we have

(4. 3) $y^i = B^i_\alpha y^\alpha$, $\partial y^i / \partial y^\alpha = B^i_\alpha$.

Also for a general tangent vector X^i (or X^α) to M_{n-1}, we have

(4. 4) $X^i = B^i_\alpha X^\alpha$.

The fundamental function $\overline{L}(u^\alpha, y^\alpha)$ on M_{n-1} induced from $L(x^i, y^i)$ of M_n is given by

(4. 5) $\overline{L}(u^\alpha, y^\alpha) = L(x^i(u^\alpha), B^i_\alpha y^\alpha)$.

Then the r-metric tensor $g_{\alpha\beta}(u^\alpha, y^\alpha) = \frac{1}{2}\partial^2 \overline{L}^2 / \partial y^\alpha \partial y^\beta$ is expressible in

(4. 6) $g_{\alpha\beta}(u^\alpha, y^\alpha) = g_{ij}(x^i, y^i) B^i_\alpha B^j_\beta$,

where g_{ij} is the r-metric tensor on M_n.

The covariant vector y_α corresponding to y^α is also expressible in

(4. 7) $y_\alpha = g_{\alpha\beta} y^\beta = \overline{L}\partial\overline{L} / \partial y^\alpha = y_i B^i_\alpha$.

With respect to the tensor g_{ij}, we can choose a unit normal vector $N^i(u^\alpha, y^\alpha)$ at each point (u^α) of M_{n-1} such that

(4. 8) $g_{ij} N^i B^j_\alpha = 0$, $g_{ij} N^i N^j = 1$.

Let (B^α_i, N_i) be the inverse matrix of (B^i_α, N^i). Then the following relations hold:

(4. 9) $B^i_\alpha B^\beta_i = \delta^\beta_\alpha$, $B^i_\alpha B^\alpha_j = \delta^i_j - N^i N_j$, $N^i y_i = N_j y^j = 0$,

$$(4.10)\qquad g^{\alpha\beta}=g^{ij}B^{\alpha}_{i}B^{\beta}_{j},\ B^{\alpha}_{i}=g^{\alpha\beta}g_{ij}B^{j}_{\beta},$$

$$(4.11)\qquad g_{ij}B^{j}_{\alpha}=g_{\alpha\beta}B^{\beta}_{i},\ g^{\alpha\beta}B^{i}_{\beta}=g^{ij}B^{\alpha}_{j},\ N_{i}=g_{ij}N^{j},$$

where $g^{\alpha\beta}$ and g^{ij} are the reciprocal tensors of $g_{\alpha\beta}$ and g_{ij} respectively.

Hereafter we shall use the following notation:

$$(4.12)\qquad \begin{aligned}&B^{ij}_{\alpha\beta}=B^{i}_{\alpha}B^{j}_{\beta},\ B^{ijk}_{\alpha\beta\gamma}=B^{i}_{\alpha}B^{j}_{\beta}B^{k}_{\gamma},\ B^{\alpha jk}_{i\beta\gamma}=B^{\alpha}_{i}B^{j}_{\beta}B^{k}_{\gamma},\\ &B^{\alpha\beta k}_{ij\gamma}=B^{\alpha}_{i}B^{\beta}_{j}B^{k}_{\gamma}.\end{aligned}$$

Now we define $\mu_{\alpha\beta}$, μ_{α} and μ as follows:

$$(4.13)\qquad \mu_{\alpha\beta}=C_{ijk}B^{ij}_{\alpha\beta}N^{k},\ \mu_{\alpha}=C_{ijk}B^{i}_{\alpha}N^{j}N^{k},\ \mu=C_{ijk}N^{i}N^{j}N^{k}.$$

Then the following relations hold [3]:

$$(4.14)\qquad \begin{aligned}&\partial B^{\alpha}_{i}/\partial y^{\beta}=B^{\alpha}_{i\|\beta}=2\mu^{\alpha}_{\beta}N_{i},\ N_{i\|\beta}=\mu_{\beta}N_{i},\\ &N^{i}_{\|\beta}=-2\mu^{\alpha}_{\beta}B^{i}_{\alpha}-\mu_{\beta}N^{i},\ \mu^{\alpha}_{\beta}=g^{\alpha\gamma}\mu_{\gamma\beta}.\end{aligned}$$

From (4. 6), (4. 10), (4. 13) and (4. 14) we have

$$(4.15)\qquad g_{\alpha\beta\|\gamma}=2C_{\alpha\beta\gamma}=2C_{ijk}B^{ijk}_{\alpha\beta\gamma},\ g^{\alpha\beta}_{\ \ \|\gamma}=-2C^{\alpha}_{\varepsilon\gamma}g^{\varepsilon\beta}=-2C^{i}_{rk}g^{rj}B^{\alpha\beta k}_{ij\gamma}.$$

§5. Induced connections.

Let vectors y^{i} and X^{i} be related by (4. 3) and (4. 4) respectively. Then the expressions (1. 4), (1. 5) and (3. 2) are written in

$$(5.1)\qquad \begin{aligned}&Dy^{i}=(B^{i}_{0\gamma}+\Gamma^{i}_{k}B^{k}_{\gamma})du^{\gamma}+B^{i}_{\beta}dy^{\beta},\ B^{i}_{\beta\gamma}=\partial B^{i}_{\beta}/\partial u^{\gamma},\\ &B^{i}_{0\gamma}=B^{i}_{\beta\gamma}y^{\beta},\end{aligned}$$

$$(5.2)\qquad \begin{aligned}DX^{i}=B^{i}_{\beta}dX^{\beta}+(B^{i}_{\beta\gamma}&+\Gamma^{i}_{jk}B^{jk}_{\beta\gamma}+C^{i}_{jh}\Gamma^{h}_{k}B^{jk}_{\beta\gamma}\\ &+C^{i}_{jk}B^{j}_{\beta}B^{k}_{0\gamma})X^{\beta}du^{\gamma}+C^{i}_{jk}B^{jk}_{\beta\gamma}X^{\beta}dy^{\gamma},\end{aligned}$$

$$(5.3)\qquad \overset{0}{D}X^{i}=B^{i}_{\beta}dX^{\beta}+(B^{i}_{\beta\gamma}+\Gamma^{i}_{jk}B^{jk}_{\beta\gamma})X^{\beta}du^{\gamma}.$$

Now we define Dy^{α}, DX^{α} and $\overset{0}{D}X^{\alpha}$ as follows:

$$(5.4)\qquad Dy^{\alpha}=B^{\alpha}_{i}Dy^{i},\quad DX^{\alpha}=B^{\alpha}_{i}DX^{i},\ \overset{0}{D}X^{\alpha}=B^{\alpha}_{i}\overset{0}{D}X^{i}.$$

Then on M_{n-1} we can define a connection $\overline{\Gamma}=(\Gamma^{\alpha}_{\beta\gamma},\ \Gamma^{\alpha}_{\gamma},\ C^{\alpha}_{\beta\gamma})$ or $\overline{\Gamma}_0=(\overset{0}{\Gamma}{}^{\alpha}_{\beta\gamma},\ \Gamma^{\alpha}_{\gamma},\ 0)$ by means of (5. 4). In terms of these connections, Dy^{α}, DX^{α} and $\overset{0}{D}X^{\alpha}$ are expressible as follows:

$$(5.\ 5)\qquad Dy^{\alpha}=dy^{\alpha}+\Gamma^{\alpha}_{\gamma}du^{\gamma},$$

$$(5.\ 6)\qquad \begin{aligned} DX^{\alpha}&=dX^{\alpha}+(\Gamma^{\alpha}_{\beta\gamma}+C^{\alpha}_{\beta\sigma}\Gamma^{\sigma}_{\gamma})du^{\gamma}+C^{\alpha}_{\beta\gamma}X^{\beta}dy^{\gamma}\\ &=X^{\alpha}_{\ |\gamma}du^{\gamma}+X^{\alpha}|_{\gamma}Dy^{\gamma},\end{aligned}$$

$$(5.\ 6)_1\qquad X^{\alpha}_{\ |\gamma}=\partial X^{\alpha}/\partial u^{\gamma}-\Gamma^{\beta}_{\gamma}\partial X^{\alpha}/\partial y^{\beta}+\Gamma^{\alpha}_{\beta\gamma}X^{\beta},$$

$$(5.\ 6)_2\qquad X^{\alpha}|_{\gamma}=\partial X^{\alpha}/\partial y^{\gamma}+C^{\alpha}_{\beta\gamma}X^{\beta},$$

$$(5.\ 7)\qquad \overset{0}{D}X^{\alpha}=dx^{\alpha}+\overset{0}{\Gamma}{}^{\alpha}_{\beta\gamma}X^{\beta}du^{\gamma}=X^{\alpha}_{\ ;\gamma}du^{\gamma}+X^{\alpha}_{\ \|\gamma}Dy^{\gamma},$$

$$(5.\ 7)_1\qquad X^{\alpha}_{\ ;\gamma}=\partial X^{\alpha}/\partial u^{\gamma}-\Gamma^{\beta}_{\gamma}\partial X^{\alpha}/\partial y^{\beta}+\overset{0}{\Gamma}{}^{\alpha}_{\beta\gamma}X^{\beta},\quad X^{\alpha}_{\ \|\gamma}=\partial X^{\alpha}/\partial y^{\gamma}.$$

If we first substitute (5. 1) in (5. 4) and compare the result with (5. 5), then we have

$$(5.\ 8)\qquad \Gamma^{\alpha}_{\gamma}=B^{\alpha}_{i}(B^{i}_{0\gamma}+\Gamma^{i}_{k}B^{k}_{\gamma}).$$

Similarly from (5. 2), (5. 4) and (5. 6) we obtain

$$(5.\ 9)\qquad \Gamma^{\alpha}_{\beta\gamma}=B^{\alpha}_{i}\{B^{i}_{\beta\gamma}+\Gamma^{i}_{j\,k}B^{jk}_{\beta\gamma}+B^{j}_{\beta}C^{i}_{j\,k}(B^{k}_{0\gamma}-\Gamma^{\varepsilon}_{\gamma}B^{k}_{\varepsilon}+\Gamma^{k}_{h}B^{h}_{\gamma})\},$$

$$(5.\ 10)\qquad C^{\alpha}_{\beta\gamma}=C^{i}_{j\,k}B^{\alpha jk}_{i\beta\gamma}.$$

Next substituting (5. 8) in (5. 9) and making use of (4. 9) we have

$$(5.\ 11)\qquad \Gamma^{\alpha}_{\beta\gamma}=B^{\alpha}_{h}(B^{h}_{\beta\gamma}+\Gamma^{h}_{j\,k}B^{jk}_{\beta\gamma}+C^{h}_{\alpha k}N^{k}H_{\gamma}),\quad C^{i}_{\beta k}=C^{i}_{j\,k}B^{j}_{\beta},$$

$$(5.\ 11)_1\qquad H_{\gamma}=N_{j}(B^{j}_{0\gamma}+\Gamma^{j}_{h}B^{h}_{\gamma}).$$

Contracting (5. 11) by B^{i}_{α}, from (4. 9) we have

$$(5.\ 12)\qquad B^{i}_{\beta\gamma}+\Gamma^{i}_{j\,k}B^{jk}_{\beta\gamma}+C^{i}_{\beta k}N^{k}H_{\gamma}=\Gamma^{\alpha}_{\beta\gamma}B^{i}_{\alpha}+H_{\beta\gamma}N^{i},$$

$$(5.\ 12)_1\qquad H_{\beta\gamma}=N_{i}(B^{i}_{\beta\gamma}+\Gamma^{i}_{j\,k}B^{jk}_{\beta\gamma}+C^{i}_{\beta k}N^{k}H_{\gamma}).$$

We shall call a connection $\overline{\Gamma}=(\Gamma^{\alpha}_{\beta\gamma},\ \Gamma^{\alpha}_{\gamma},\ C^{\alpha}_{\beta\gamma})$ on M_{n-1} obtained as above the *induced TM-connection* and denote it by *ITMΓ*. In this case, H_{γ}

is the normal curvature vector, while $H_{\beta\gamma}$ is the second fundamental tensor.

Lastly from (5. 3), (5. 4) and (5. 7) we obtain

$$\overset{0}{\Gamma}{}^{\alpha}_{\beta\gamma} = B^{\alpha}_{h} (B^{h}_{\beta\gamma} + \Gamma^{h}_{j\,k} B^{jk}_{\beta\gamma}), \tag{5.13}$$

$$B^{i}_{\beta\gamma} + \Gamma^{i}_{j\,k} B^{jk}_{\beta\gamma} = \overset{0}{\Gamma}{}^{\alpha}_{\beta\gamma} B^{i}_{\alpha} + \overset{0}{H}_{\beta\gamma} N^{i}, \quad \overset{0}{H}_{\beta\gamma} = N_{i} (B^{i}_{\beta\gamma} + \Gamma^{i}_{j\,k} B^{jk}_{\beta\gamma}). \tag{5.14}$$

We shall call a connection $\overline{\Gamma}_0 = (\overset{0}{\Gamma}{}^{\alpha}_{\beta\gamma}, \Gamma^{\alpha}_{\gamma}, 0)$ on M_{n-1} the *induced T-M*(0) *-connection* and denote it by *ITM* (0). In this case, $\overset{0}{H}_{\beta\gamma}$ is the second fundamental tensor for $\overline{\Gamma}_0$.

Note 5. 1. The induced non-linear connection Γ^{α}_{γ} is common for both of $\overline{\Gamma}$ and $\overline{\Gamma}_0$.

Contracting (5. 11) and (5. 13) by y^{β}, from (5. 8) we have

$$y^{\beta} \Gamma^{\alpha}_{\beta\gamma} = y^{\beta} \overset{0}{\Gamma}{}^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\gamma}. \tag{5.15}$$

Similarly from (5. 12)$_1$ and (5. 14) we obtain

$$H_{0\gamma} = \overset{0}{H}_{0\gamma} = H_{\gamma}, \quad H_{0} = N^{i} (B^{i}_{0\,0} + 2G^{i}). \tag{5.16}$$

Note 5. 2. If the original connection Γ (or $\Gamma(0)$) is a *WTM* (or *WT-M*(0))-connection, then the connection on M_{n-1} induced from Γ(or $\Gamma(0)$) will be called the *induced WTM* (or *WTM*(0))*-connection*. In this case, it should be noticed that H_0 is rewritten in

$$H_{0} = N_{i} (B^{i}_{0\,0} + 2G^{i} + T^{i}_{0}). \tag{5.16$_1$}$$

Differentiating (5. 11)$_1$ by y^{β} and making use of (4. 14) and (5. 14), we have

$$\begin{aligned} H_{\gamma\|\beta} &= H_{\beta\gamma} - Q^{n}_{\beta\gamma} \\ &= \overset{0}{H}_{\beta\gamma} + \mu_{\beta} H_{\gamma} - Q^{n}_{\beta\gamma}, \quad Q^{n}_{\beta\gamma} = Q^{i}_{j\,k} N_{i} B^{jk}_{\beta\gamma}. \end{aligned} \tag{5.17}$$

Suppose that $H_{\gamma} = 0$. Then it follows from (5. 17) that

$$H_{\beta\gamma} = Q^{n}_{\beta\gamma} \quad \text{or} \quad \overset{0}{H}_{\beta\gamma} = Q^{n}_{\beta\gamma}. \tag{5.17$_1$}$$

Conversely we assume that (5. 17)$_1$ holds. Then if we contract (5. 17) by y^{β}, then we have $H_{\gamma} = 0$. Consequently we can state

Lemma 1. *The normal curvature vector H_γ vanishes if and only if the second fundamental tensor is expressible in* (5. 17)$_1$.

Let $X^{i\alpha}_{j\beta}$ be an object defined on M_{n-1} such that it is a tensor in M_n of type (1, 1) and, at the same time, a tensor in M_n of type (1, 1). Then the relative h- and v-covariant derivatives are defined as follows [**3**]:

(5. 18) $$X^{i\alpha}_{j\beta|\gamma} = \delta_\gamma X^{i\alpha}_{j\beta} + X^{k\alpha}_{j\beta}\Gamma^{i}_{k\gamma} - X^{i\alpha}_{k\beta}\Gamma^{k}_{j\gamma} + X^{i\sigma}_{j\beta}\Gamma^{\alpha}_{\sigma\gamma} - X^{i\alpha}_{j\sigma}\Gamma^{\sigma}_{\beta\gamma},$$

where $\delta_\gamma = \partial/\partial u^\gamma - \Gamma^\sigma_\gamma \partial/\partial y^\sigma$ and $\Gamma^{i}_{k\gamma} = \Gamma^{i}_{kj} B^j_\gamma + C^{i}_{kj} N^j H_\gamma$.

(5. 19) $$X^{i\alpha}_{j\beta}|_\gamma = X^{i\alpha}_{j\beta\|\gamma} + X^{k\alpha}_{j\beta} C^{i}_{k\gamma} - X^{i\alpha}_{k\beta} C^{k}_{j\gamma} + X^{i\sigma}_{j\beta} C^{\alpha}_{\sigma\gamma} - X^{i\alpha}_{j\sigma} C^{\sigma}_{\beta\gamma},$$

where $C^{k}_{j\gamma} = C^{k}_{ji} B^i_\gamma$.

(5. 20) $$X^{i\alpha}_{j\beta:\gamma} = \delta_\gamma X^{i\alpha}_{j\beta} + X^{k\alpha}_{j\beta}\overset{0}{\Gamma}{}^{i}_{k\gamma} - X^{i\alpha}_{k\beta}\overset{0}{\Gamma}{}^{k}_{j\gamma} + X^{i\sigma}_{j\beta}\overset{0}{\Gamma}{}^{\alpha}_{\sigma\gamma} - X^{i\alpha}_{j\sigma}\overset{0}{\Gamma}{}^{\sigma}_{\beta\gamma},$$

where $\Gamma^{i}_{k\gamma} = \Gamma^{i}_{kj} B^j_\gamma$.

By virtue of (5. 18)～(5. 20) we can derive

(5. 21) $$B^i_{\alpha|\beta} = H_{\alpha\beta} N^i \ , \ B^i_{\alpha:\beta} = \overset{0}{H}_{\alpha\beta} N^i \ , \ B^i_\alpha|_\beta = \mu_{\alpha\beta} N^i \ .$$

If we contract (5. 12) (or (5. 14)) by y^β and make use of (5. 15) and (5. 16), then we have

$$B^i_{0\gamma} + \Gamma^i_k B^k_\gamma = \Gamma^\alpha_\gamma B^i_\alpha + H_\gamma N^i \ ,$$

which implies

(5. 22) $$\delta_\gamma = B^k_\gamma \partial/\partial x^k + (B^i_{0\gamma} - \Gamma^\alpha_\gamma B^i_\gamma)\partial/\partial y^i = B^i_\gamma \delta_i + N^i H_\gamma \partial/\partial y^i,$$

where $\delta_i = \partial/\partial x^i - \Gamma^j_i \partial/\partial y^j$. Then by virtue of (5. 22) we obtain $\overline{L}_{|\gamma} = \overline{L}_{:\gamma} = B^i_\alpha L_{|i} = 0$. Hence we can state

Lemma 2. *The induced TM-connection and TM*(0) *-connection are both metrical.*

From (4. 6), (5. 18), (5. 19), (5. 21) and (5. 22) we have

(5. 23) $$g_{\alpha\beta|\gamma} = g_{ij|k} B^{ijk}_{\alpha\beta\gamma} \ , \ g_{\alpha\beta}|_\gamma = g_{ij}|_k B^{ijk}_{\alpha\beta\gamma} = 0.$$

Similarly we obtain

(5. 24) $$g_{\alpha\beta:\gamma} = g_{ij|k} B^{ijk}_{\alpha\beta\gamma} + 2\mu_{\alpha\beta} H_\gamma \ , \ g_{\alpha\beta\|\gamma} = 2C_{ijk} B^{ijk}_{\alpha\beta\gamma} \ .$$

Differentiating H_0 in (5. 16) by y^a, from (4. 14) we have

$$H_{0\|\alpha} = H_0 \mu_\alpha + 2\overset{h}{H}_\alpha, \quad \overset{b}{H}_\alpha = N_i (B^i_{0\alpha} + G^i_j B^j_\alpha). \tag{5. 25}$$

Note 5.3. For the induced *WTM* (or *WTM*(0))-connection, Lemma 1 and Lemma 2 are still valid. And the above expression is rewritten in

$$H_{0\|\alpha} = H_0 \mu_\alpha + 2\overset{b}{H}_\alpha + (T^i_{j0} + T^i_j) N_i B^j_\alpha. \tag{5. 25$_1$}$$

For the later use we define two tensors on M_{n-1} as follows:

$$v_{\alpha\beta\gamma} \overset{def}{=} \mu_{\alpha\beta\|\gamma} + 2C_{\alpha\beta\sigma}\mu^\sigma_\gamma + \mu_{\alpha\beta}\mu_\gamma, \tag{5. 26}$$

$$\mu^\alpha_{\beta\gamma} \overset{def}{=} \mu^\alpha_{\beta\|\gamma} + \mu^\alpha_\beta \mu_\gamma. \tag{5. 27}$$

Then we can state

Lemma 3. *The tensor $v_{\alpha\beta\gamma}$ is symmetric in all indices, while the tensor $\mu^\alpha_{\beta\gamma}$ is symmetric in lower indices.*

Proof. From (4. 14) and (5. 26) we first have

$$B^{ijk}_{\alpha\beta\gamma} C_{ijk\|h} N^h = B^{ijk}_{\alpha\beta\gamma} C_{ijh\|k} N^h = B^{ij}_{\alpha\beta} C_{ijh\|\gamma} N^h$$

$$= \mu_{\alpha\beta\|\gamma} - B^{ij}_{\alpha\beta} C_{ijh} N^h_{\|\gamma} = v_{\alpha\beta\gamma},$$

which shows that the tensor $v_{\alpha\beta\gamma}$ is symmetric in α, β and γ because of the symmetry of C_{ijk}. Next, from (5. 26) and (5. 27) we have

$$g^{\varepsilon\alpha} v_{\varepsilon\beta\gamma} - 2(C^\alpha_{\gamma\sigma}\mu^\sigma_\beta + C^\alpha_{\beta\sigma}\mu^\sigma_\gamma) = \mu^\alpha_{\beta\gamma},$$

whict indicates that the tensor $\mu^\alpha_{\beta\gamma}$ is symmetric in β and γ. Q.E.D.

§6. The induced *TM*(or *TM*(0))-and *WTM*(or *WTM*(0))-connections. Let us first consider the induced *TM*-connection *ITMΓ*. Then we can state

Lemma 4. *The ITMΓ satisfies all the axioms* **(TM1)**～**(TM6)**.

Proof. By virtue of Lemma 2 and (5. 15), the axioms **(TM1)** and **(TM2)** hold. Next it is seen from (5. 10) and (5. 23) that **(TM3)** and **(TM4)** hold. Since (5. 8) and (5. 15) implies $y^\alpha_{|\gamma}=0$, it follows from (5. 23) that $y^\alpha D g_{\alpha\beta}=0$, that is, **(TM5)′** holds. Lastly we shall prove that **(TM6)** holds. If we denote the Christoffel symbols of the second kind formed with $g_{\alpha\beta}$ by

$\gamma_{\beta}{}^{\alpha}{}_{\gamma}$, then we have **[4]**

$$\gamma_{\alpha}{}^{\gamma}{}_{\beta}=B^{\gamma}{}_{i}(B_{\alpha}{}^{i}{}_{\beta}+\gamma_{j}{}^{i}{}_{k}B^{jk}_{\alpha\beta})+g^{\gamma\sigma}C_{ijk}(B^{ij}_{\beta\sigma}B^{k}_{0\alpha}+B^{ij}_{\sigma\alpha}B^{k}_{0\beta}-B^{ij}_{\alpha\beta}B^{k\sigma}), \tag{6. 1}$$

where $\gamma_{j}{}^{i}{}_{k}$ are the Christoffel symbols of the second kind formed with g_{ij}.

Contracting (6. 1) by $y^{\alpha}y^{\beta}$, we have

$$2G^{\gamma}=\gamma_{\alpha}{}^{\gamma}{}_{\beta}y^{\alpha}y^{\beta}=B^{\gamma}{}_{i}(B_{0}{}^{i}{}_{0}+2G^{i}). \tag{6. 2}$$

On the other hand, if we contract (5. 8) by y^{γ} and replace the index α with γ then we have

$$\Gamma^{\gamma}{}_{0}=B^{\gamma}{}_{i}(B_{0}{}^{i}{}_{0}+2G^{i}). \tag{6. 3}$$

Therefore from (6. 2) and (6. 3) we can conclude that (**TM6**) holds.

Q. E. D.

We put

$$\overset{b}{\Gamma}{}^{\alpha}{}_{\gamma}=B^{\alpha}{}_{i}(B_{0}{}^{i}{}_{\gamma}+G^{i}{}_{k}B^{k}{}_{\gamma}),\ T^{\alpha}{}_{\gamma}=T^{i}{}_{k}B^{\alpha k}_{i\gamma},\ Q_{\beta}{}^{\alpha}{}_{\gamma}=Q_{j}{}^{i}{}_{k}B^{\alpha jk}_{i\beta\gamma}. \tag{6. 4}$$

Differentiating (6. 2) and making use of (4. 14), (5. 16) and (6. 4), we have

$$G^{\alpha}{}_{\gamma}=G^{\alpha}{}_{\|\gamma}=\mu^{\alpha}{}_{\gamma}H_{0}+\overset{b}{\Gamma}{}^{\alpha}{}_{\gamma}. \tag{6. 5}$$

If we subtract (6. 5) from (5. 8), then we obtain

$$\widetilde{T}^{\alpha}{}_{\gamma}=\Gamma^{\alpha}{}_{\gamma}-G^{\alpha}{}_{\gamma}=-\mu^{\alpha}{}_{\gamma}H_{0}+T^{\alpha}{}_{\gamma}. \tag{6. 6}$$

Differentiating (5. 8) by y^{β}, from (4. 14) and (5. 11)$_1$ we have

$$\Gamma^{\alpha}{}_{\gamma\|\beta}=2\mu^{\alpha}{}_{\beta}H_{\gamma}+B^{\alpha}{}_{i}(B_{\beta}{}^{i}{}_{\gamma}+\Gamma^{i}{}_{k\|j}B^{jk}_{\beta\gamma}). \tag{6. 7}$$

Subtracting (6. 7) from (5. 11), from (6. 4) we obtain

$$\widetilde{Q}_{\beta}{}^{\alpha}{}_{\gamma}=\Gamma_{\beta}{}^{\alpha}{}_{\gamma}-\Gamma^{\alpha}{}_{\gamma\|\beta}=-\mu^{\alpha}{}_{\beta}H_{\gamma}+Q_{\beta}{}^{\alpha}{}_{\gamma}. \tag{6. 8}$$

From (5. 11) and (6. 8) we have

$$\widetilde{\tau}_{\beta}{}^{\alpha}{}_{\gamma}=\tau_{j}{}^{i}{}_{k}B^{\alpha jk}_{i\beta\gamma}+(\mu^{\alpha}{}_{\beta}H_{\gamma}-\mu^{\alpha}{}_{\gamma}H_{\beta}),\ \widetilde{P}_{\beta}{}^{\alpha}{}_{\gamma}=\mu^{\alpha}{}_{\beta}H_{\gamma}-Q_{\beta}{}^{\alpha}{}_{\gamma}. \tag{6. 9}$$

Since $\mu^{0}{}_{\beta}=\mu^{\alpha}{}_{0}=0$, $T^{0}{}_{k}=T^{i}{}_{0}=0$ and $Q_{j}{}^{0}{}_{k}=Q_{0}{}^{i}{}_{k}=0$, it follows from

(6. 6) and (6. 8) that

$$(6.10)\qquad \widetilde{T}^0{}_\gamma = \widetilde{T}^\alpha{}_0 = 0,\ \widetilde{Q}_\beta{}^0{}_\gamma = \widetilde{Q}_0{}^\alpha{}_\gamma = 0.$$

Thus, from Theorem 1. 1, Corollary 1. 1. 1 and Lemma 4 we can state

Theorem 6. 1. *The induced TM-connection ITMΓ is the TM-connection on* M_{n-1} *determined by tensors* $\widetilde{T}^\alpha{}_\gamma$ *and* $\widetilde{Q}_\beta{}^\alpha{}_\gamma$ *in* (6. 6) *and* (6. 8) *respectively, and the h- and hv- torsion tensors* $\widetilde{\tau}_\beta{}^\alpha{}_\gamma$ *and* $\widetilde{P}_\beta{}^\alpha{}_\gamma$ *are given by* (6. 9).

When $T^i{}_k = 0$ and $Q_j{}^i{}_k = 0$, from (6. 6), (6. 8) and (6. 9) we have

$$(6.11)\qquad \overset{h}{T}{}^\alpha{}_\gamma = -\mu^\alpha{}_\gamma H_0,\ \overset{h}{Q}{}_\beta{}^\alpha{}_\gamma = -\mu^\alpha{}_\beta \overset{b}{H}_\gamma,$$

$$(6.12)\qquad \overset{h}{\tau}{}_\beta{}^\alpha{}_\gamma = \mu^\alpha{}_\beta \overset{b}{H}_\gamma - \mu^\alpha{}_\gamma \overset{b}{H}_\beta,\ \overset{h}{P}{}_\beta{}^\alpha{}_\gamma = \mu^\alpha{}_\beta \overset{b}{H}_\gamma .$$

From (5. 25), (5. 27), (6. 11) and Lemma 3 we have

$$(6.13)\qquad \overset{h}{T}{}_\beta{}^\alpha{}_\gamma = \overset{h}{T}{}^\alpha{}_{\gamma\|\beta} = -\mu^\alpha{}_{\beta\gamma} H_0 - 2\mu^\alpha{}_\gamma \overset{b}{H}_\beta .$$

From (6. 11) and (6. 13) we obtain

$$(6.14)\qquad \overset{h}{\Gamma}{}_\beta{}^\alpha{}_\gamma = G_\beta{}^\alpha{}_\gamma + \overset{h}{T}{}_\beta{}^\alpha{}_\gamma + \overset{h}{Q}{}_\beta{}^\alpha{}_\gamma = G_\beta{}^\alpha{}_\gamma - \mu^\alpha{}_{\beta\gamma} \overset{b}{H}_0 - 2\mu^\alpha{}_\gamma \overset{b}{H}_\beta - \mu^\alpha{}_\beta \overset{b}{H}_\gamma,$$

where $G_\beta{}^\alpha{}_\gamma$ is the intrinsic h-connection of Berwald, namely $G_\beta{}^\alpha{}_\gamma = G^\alpha{}_{\gamma\|\beta}$.

From (5. 11) and $\Gamma_j{}^i{}_k = G_j{}^i{}_k$, we have

$$(6.15)\qquad \overset{h}{\Gamma}{}_\beta{}^\alpha{}_\gamma = \overset{b}{\Gamma}{}_\beta{}^\alpha{}_\gamma + \mu^\alpha{}_\beta \overset{b}{H}_\gamma,\ \overset{b}{\Gamma}{}_\beta{}^\alpha{}_\gamma \overset{def}{=} B^\alpha{}_i\,(B^i_{\beta\gamma} + G_j{}^i{}_k B^{jk}_{\beta\gamma}).$$

Consequently by virtue of Corollary 1. 1. 2 we can state

Corollary 6. 1. 1. *The induced Hashiguchi connection IHΓ is the TM-connection on* M_{n-1} *determined by tensors* $\overset{h}{T}{}^\alpha{}_\gamma$ *and* $\overset{h}{Q}{}_\beta{}^\alpha{}_\gamma$ *in* (6. 11). *The h- and hv-torsion tensors are given* (6. 12) *and the h-connection is expressible in* (6. 14) (*or* (6. 15)).

Now, by the use of (6. 14) and (6. 15) we can calculate $G_\beta{}^\alpha{}_\gamma$ and get

$$(6.16)\qquad G_\beta{}^\alpha{}_\gamma = \overset{b}{\Gamma}{}_\beta{}^\alpha{}_\gamma + \mu^\alpha{}_{\beta\gamma} H_0 + 2(\mu^\alpha{}_\beta \overset{b}{H}_\alpha + \mu^\alpha{}_\gamma \overset{b}{H}_\beta).$$

Similarly, when $T^i{}_k = 0$ and $Q_j{}^i{}_k = -P_j{}^i{}_k$ we obtain

$$(6.17)\qquad \overset{c}{T}{}^\alpha{}_\gamma = -\mu^\alpha{}_\gamma H_0,\ \overset{c}{Q}{}_\beta{}^\alpha{}_\gamma = -\mu^\alpha{}_\beta \overset{b}{H}_\gamma - P_\beta{}^\alpha{}_\gamma,\ P_\beta{}^\alpha{}_\gamma = P_j{}^i{}_k B^{\alpha jk}_{i\beta\gamma},$$

$$(6.18)\qquad \overset{c}{\tau}{}_\beta{}^\alpha{}_\gamma = \mu^\alpha{}_\beta \overset{b}{H}_\gamma - \mu^\alpha{}_\gamma \overset{b}{H}_\beta,\ \overset{c}{P}{}_\beta{}^\alpha{}_\gamma = \mu^\alpha{}_\beta \overset{b}{H}_\gamma + P_\beta{}^\alpha{}_\gamma .$$

Since $\overset{c}{T}{}^{\alpha}{}_{\gamma} = \overset{h}{T}{}^{\alpha}{}_{\gamma}$, from (6. 13) and (6. 17) we have

(6. 19) $$\overset{c}{\Gamma}{}^{\alpha}_{\beta\gamma} = G^{\alpha}_{\beta\gamma} + \overset{c}{T}{}^{\alpha}_{\beta\gamma} + \overset{c}{Q}{}^{\alpha}_{\beta\gamma} = G^{\alpha}_{\beta\gamma} - \mu^{\alpha}{}_{\beta\gamma} H_0 - 2\mu^{\alpha}{}_{\gamma} \overset{b}{H}_{\beta} - \mu^{\alpha}{}_{\beta} \overset{b}{H}_{\gamma} - P^{\alpha}_{\beta\gamma}.$$

On the other hand, from (5. 11) and $\Gamma^{i}_{jk} = \Gamma^{*i}_{jk}$ we obtain

(6. 19)$_1$ $$\overset{c}{\Gamma}{}^{\alpha}_{\beta\gamma} = \overset{b}{\Gamma}{}^{\alpha}_{\beta\gamma} - P^{\alpha}_{\beta\gamma} + \mu^{\alpha}{}_{\beta} \overset{b}{H}_{\gamma}.$$

Consequently from Corollary 1. 1. 3 we can state

Corollary 6. 1. 2. *The induced Cartan connection ICΓ is the TM-connection on* M_{n-1} *determined by tensors* $\overset{c}{T}{}^{\alpha}{}_{\gamma}$ *and* $\overset{c}{Q}{}^{\alpha}_{\beta\gamma}$ *in* (6. 17). *The h- and hv-torsion tensors are given by* (6. 18) *and the h-connection is expressible in* (6. 19) (*or* (6. 19)$_1$).

Note 6. 1. We denote the intrinsic h-connection of Cartan by $\Gamma^{*\alpha}_{\beta\gamma}$ and the *hv*-torsion tensor with respect to $\Gamma^{*\alpha}_{\beta\gamma}$ by $P^{*\alpha}_{\beta\gamma}$. From (6. 19) and (6. 19)$_1$ we can calculate $G^{\alpha}_{\beta\gamma}$, but not $\Gamma^{*\alpha}_{\beta\gamma}$ or $P^{*\alpha}_{\beta\gamma}$.

Next we consider the induced *WTM*-connection $\overline{\Gamma}$. For this connection, it is easily seen that the $\overline{\Gamma}$ satisfies **(WTM1)**~**(WTM5)**. Hence from Theorem 1. 2 we can state

Theorem 6. 2. *The induced WTM-connection* $\overline{\Gamma}$ *is the WTM-connection on* M_{n-1} *determined by tensor* $\widetilde{T}{}^{\alpha}{}_{\gamma}$ *and* $\widetilde{Q}{}^{\alpha}_{\beta\gamma}$ *in* (6. 6) *and* (6. 8), *and the h- and hv-torsion tensors* $\widetilde{\tau}{}^{\alpha}_{\beta\gamma}$ *and* $\widetilde{P}{}^{\alpha}_{\beta\gamma}$ *are given by* (6. 9).

Note 6. 2. For the above connection $\overline{\Gamma}$, the vector $\widetilde{T}{}^{\alpha}{}_{0}$ does not vanish, that is, a path with respect to $\overline{\Gamma}$ is not, in general, a geodesic of M_{n-1}.

Now we shall consider the induced *TM*(0)-connection $\overline{\Gamma}_0$. If we take account of the definition of $\overline{\Gamma}_0$ and use the proof of Lemma 4, then we can prove that all the axioms **(T1)**~**(T5)** hold. From (5. 8), (5. 13), (6. 5) and (6. 7) we obtain

(6. 20) $$\widetilde{T}{}^{\alpha}{}_{\gamma} = -\mu^{\alpha}{}_{\gamma} H_0 + T^{\alpha}{}_{\gamma}, \quad \widetilde{Q}{}^{\alpha}_{\beta\gamma} = -2\mu^{\alpha}{}_{\beta} H_{\gamma} + Q^{\alpha}_{\beta\gamma},$$

(6. 21) $$\widetilde{\tau}{}^{\alpha}_{\beta\gamma} = \tau^{i}_{jk} B^{\alpha jk}_{i\beta\gamma}, \quad \widetilde{P}{}^{\alpha}_{\beta\gamma} = 2\mu^{\alpha}{}_{\beta} H_{\gamma} - Q^{\alpha}_{\beta\gamma}.$$

Then it is seen that tensors $\widetilde{T}{}^{\alpha}{}_{\gamma}$ and $\widetilde{Q}{}^{\alpha}_{\beta\gamma}$ in (6. 20) satisfy (a) and (b) respectively. Consequently, from Theorem 3. 1 we can state

Theorem 6. 3. *The induced TM*(0) *-connection ITMΓ*(0) *is the TM*(0)*-connection on* M_{n-1} *determined by tensors* $\widetilde{T}{}^{\alpha}{}_{\gamma}$ *and* $\widetilde{Q}{}^{\alpha}_{\beta\gamma}$ *in* (6. 20), *and the h-*

and hv-torsion tensors $\widetilde{\tau}^{\alpha}_{\beta\gamma}$ *and* $\widetilde{P}^{\alpha}_{\beta\gamma}$ *are given by* (6. 21).

Immediately from Corollary 3. 1. 1 we have

Corollary 6. 3. 1. *The induced* $WTM(0)$*-connection is the* $WTM(0)$*-connection on* M_{n-1} *determined by tensors* $\widetilde{T}^{\alpha}{}_{\gamma}$ *and* $\widetilde{Q}^{\alpha}_{\beta\gamma}$ *in* (6. 20). *Also in this case, Note* 6. 2 *is valid. The h- and hv-torsion tensor* $\widetilde{\tau}^{\alpha}_{\beta\gamma}$ *and* $\widetilde{P}^{\alpha}_{\beta\gamma}$ *are given by* (6. 21).

If $\tau_j{}^i{}_k = 0$, then from (6 . 21) we have $\widetilde{\tau}^{\alpha}_{\beta\gamma} = 0$. Hence we can state

Lemma 5. *If a* $TM(0)$ (*or* $WTM(0)$)*-connection is symmetric, then the induced* $TM(0)$ (*or* $WTM(0)$) *-connection is also symmetric.*

Let us consider the induced $STM(0)$-connection $\overline{\Gamma}_0$, that is, the connection on M_{n-1} induced from an $STM(0)$-connection on M_n. Then this $\overline{\Gamma}_0$ satisfies **(T1)**, **(T2)**, **(T3)**, **(T5)** and **(STM6)**, while the tensor $\widetilde{Q}^{\alpha}_{\beta\gamma}$ in(6. 20) satisfies (b).

Since $T^i{}_k = \frac{1}{2} Q^i_{k0}$ on M_n, we have $T^{\alpha}{}_{\gamma} = \frac{1}{2} Q^{\alpha}_{\gamma 0}$ on M_{n-1}. Therefore by the use of this relation, we have

(6. 22) $$\tfrac{1}{2}\widetilde{Q}^{\alpha}_{\gamma 0} = -\mu^{\alpha}{}_{\gamma} H_0 + \tfrac{1}{2} Q^{\alpha}_{\gamma 0} = -\mu^{\alpha}{}_{\gamma} H_0 + T^{\alpha}{}_{\gamma} = \widetilde{T}^{\alpha}{}_{\gamma},$$

which shows that the relation between $\widetilde{T}^{\alpha}{}_{\gamma}$ and $\widetilde{Q}^{\alpha}_{\beta\gamma}$ in (6. 20) is compatible with that in (2. 16) (or (2. 18)). Hence from (2. 17) we have

(6. 23) $$\overset{0}{\Gamma}{}^{\alpha}_{\beta\gamma} = G^{\alpha}_{\beta\gamma} + \tfrac{1}{2}(\widetilde{Q}^{\alpha}_{\beta\gamma} + \widetilde{Q}^{\alpha}_{\beta\beta}) + \tfrac{1}{4}(\widetilde{Q}^{\alpha}_{\gamma 0\|\beta} + \widetilde{Q}^{\alpha}_{\beta 0\|\gamma}).$$

Consequently, from Theorem 3. 2 and Lemma 5 we can state

Theorem 6. 4. *The induced* $STM(0)$ *-connection is the* $STM(0)$ *-connection on* M_{n-1} *determined by the tensor* $\widetilde{Q}^{\alpha}_{\beta\gamma}$ *in* (6. 20) *and the h-connection is expressible in* (6. 23).

When $Q_j{}^i{}_k = 0$, from (6. 20) and (6. 22) we have

(6. 24) $$\overset{b}{T}{}^{\alpha}{}_{\gamma} = -\mu^{\alpha}{}_{\gamma} H_0, \quad \overset{b}{Q}{}^{\alpha}_{\beta\gamma} = (-\overset{b}{P}{}^{\alpha}_{\beta\gamma}) = -2\mu^{\alpha}{}_{\beta} \overset{b}{H}_{\gamma}, \quad \overset{b}{Q}{}^{\alpha}_{\gamma 0} = -2\mu^{\alpha}{}_{\gamma} H_0.$$

From (6. 13), (6. 23) and (6. 24) we obtain

(6. 25) $$\overset{b}{\Gamma}{}^{\alpha}_{\beta\gamma} = G^{\alpha}_{\beta\gamma} - \mu^{\alpha}{}_{\beta\gamma} H_0 - 2(\mu^{\alpha}{}_{\beta} \overset{b}{H}_{\gamma} + \mu^{\alpha}{}_{\gamma} \overset{b}{H}_{\beta}).$$

Consequently from Note 3. 2 we can state

Corollary 6. 4. 1. *The induced Berwald connection* $IB\Gamma$ *is the* $STM(0)$-

connection determined by the tensor $\overset{b}{Q}{}^{\alpha}_{\beta\gamma}$ *in* (6. 24) *and the h-connection is expressible* (6. 25).

Note 6.3. The expressions (6. 16) and (6. 25) are mutually equivalent. Similarly, when $Q_j{}^i{}_k = -P_j{}^i{}_k$ we obtain

$$\overset{r}{T}{}^{\alpha}{}_{\gamma} = -\mu^{\alpha}{}_{\gamma} H_0 , \quad \overset{r}{Q}{}^{\alpha}_{\beta\gamma} = (-\overset{r}{P}{}^{\alpha}_{\beta\gamma}) = -2\mu^{\alpha}{}_{\beta} \overset{b}{H}_{\gamma} - P^{\alpha}_{\beta\gamma} ,$$
(6. 26)
$$\overset{r}{Q}{}^{\alpha}_{\gamma 0} = -2\mu^{\alpha}{}_{\gamma} H_0 .$$

(6. 27) $$\overset{r}{\Gamma}{}^{\alpha}_{\beta\gamma} = G^{\alpha}_{\beta\gamma} - \mu^{\alpha}{}_{\beta\gamma} H_0 - P^{\alpha}_{\beta\gamma} - 2(\mu^{\alpha}{}_{\beta} \overset{b}{H}_{\gamma} + \mu^{\alpha}{}_{\gamma} \overset{b}{H}_{\beta}).$$

Hence we have

Corollary 6. 4. 2. *The induced Rund connection IRΓ is the STM*(0)*-connection determined by the tensor* $\overset{r}{Q}{}^{\alpha}_{\beta\gamma}$ *in* (6. 26) *and the h-connection is expressible in* (6. 27).

Note 6.4. From Corollary 2. 1. 1 and Theorem 3. 3 we can express (6. 27) in another form (containing $\Gamma^{*\alpha}_{\beta\gamma}$).

§7. Various induced connections. In this section, we shall treat various induced connections. First we can state

Lemma 6. *If a TM-connection is r-metrical,* i. e., $Dg_{ij} = 0$, *then the induced TM-connection is also r-metrical.*

Proof. If $Dg_{ij} = 0$, then from (5. 23) we have $Dg_{\alpha\beta} = 0$. Q. E. D.

The connection on M_{n-1} induced from an *RTM*-connection on M_n is called the *induced RTM-connection.* Then taking account of Lemma 6, we can easily prove that all the axioms **(RTM1)**~**(RTM5)** hold for this connection. From (6. 6) and (6. 8) we have

(7. 1) $$\widetilde{T}{}^{\alpha}{}_{\gamma} = -\mu^{\alpha}{}_{\gamma} H_0 + T^{\alpha}{}_{\gamma} , \quad \widetilde{Q}{}^{\alpha}_{\beta\gamma} = -\mu^{\alpha}{}_{\beta} H_{\gamma} + Q^{\alpha}_{\beta\gamma} .$$

If we put $\widetilde{Z}_{\alpha\beta\gamma} = \widetilde{Q}_{\alpha\beta\gamma} - \widetilde{Q}_{\beta\alpha\gamma}$, then from (7. 1) we have

(7. 2) $$\widetilde{Z}_{\alpha\beta\gamma} = Q_{\alpha\beta\gamma} - Q_{\beta\alpha\gamma} = Z_{ijk} B^{ijk}_{\alpha\beta\gamma} ,$$

which implies that $\widetilde{Z}_{\alpha\beta\gamma} + \widetilde{Z}_{\beta\alpha\gamma} = 0$ and $\widetilde{Z}_{0\beta\gamma} = \widetilde{Z}_{\alpha 0\gamma} = 0$. Therefore from (2. 4) we have

(7. 3) $$\Gamma^{\alpha}_{\beta\gamma} = \Gamma^{*\alpha}_{\beta\gamma} - C^{\alpha}_{\beta\varepsilon} \widetilde{T}{}^{\varepsilon}{}_{\gamma} + \tfrac{1}{2} g^{\alpha\varepsilon} (\widetilde{Z}_{\beta\varepsilon\gamma} + \widetilde{T}_{\beta\varepsilon\gamma} - \widetilde{T}_{\varepsilon\beta\gamma}).$$

Consequently from 2.1 and Corollary 2.1.1 we can state

Theorem 7.1. *The induced RTM-connection is the RTM-connection on M_{n-1} determined by tensors $\widetilde{T}^{\alpha}{}_{\gamma}$ and $\widetilde{Z}_{\alpha\beta\gamma}$ in the expressions* (7.1) *and* (7.2) *respectively, and the h-connection is expressible in* (7.3).

When $T^{i}{}_{k}=0$ and $Z_{ijk}=0$, from (7.1) and (7.2) we have

$$\overset{c}{T}{}^{\alpha}{}_{\gamma}=-\mu^{\alpha}{}_{\gamma}H_{0},\ \overset{c}{Z}_{\alpha\beta\gamma}=0. \tag{7.4}$$

Since $\overset{c}{T}{}^{\alpha}{}_{\gamma}=\overset{h}{T}{}^{\alpha}{}_{\gamma}$, from (5.27) and (6.13) we have

$$\overset{c}{T}_{\varepsilon\beta\gamma}=\{-v_{\beta\gamma\varepsilon}+2(C_{\beta\gamma\sigma}\mu^{\sigma\varepsilon}+C_{\beta\varepsilon\sigma}\mu^{\sigma}{}_{\gamma})\}H_{0}-2\mu_{\beta\gamma}\overset{b}{H}_{\varepsilon}. \tag{7.5}$$

If we apply (7.4) and (7.5) to (7.3) and take account of Lemma 3, then we have

$$\overset{c}{\Gamma}{}^{\alpha}_{\beta\gamma}=\Gamma^{*\alpha}_{\beta\gamma}+H_{0}(C^{\alpha}_{\beta\varepsilon}\mu^{\varepsilon}{}_{\gamma}+C^{\alpha}_{\gamma\varepsilon}\mu^{\varepsilon}{}_{\beta}-C^{\varepsilon}_{\beta\gamma}\mu^{\alpha}{}_{\varepsilon})+\mu_{\beta\gamma}\mu^{\alpha}{}_{\varepsilon}) \\ +\mu_{\beta\gamma}\overset{b}{H}{}^{\alpha}-\mu^{\alpha}{}_{\gamma}\overset{b}{H}_{\beta}, \tag{7.6}$$

where $\overset{b}{H}{}^{\alpha}=g^{\alpha\varepsilon}\overset{b}{H}_{\varepsilon}$.

Then by means of (6.16) and (7.6) we can calculate $P^{*\alpha}_{\beta\gamma}$ and get

$$P^{*\alpha}_{\beta\gamma}=P^{\alpha}_{\beta\gamma}+H_{0}(\mu^{\alpha}{}_{\beta\gamma}+C^{\alpha}_{\beta\varepsilon}\mu^{\varepsilon}{}_{\gamma}+C^{\alpha}_{\gamma\varepsilon}\mu^{\varepsilon}{}_{\beta}-C^{\varepsilon}_{\beta\gamma}\mu^{\alpha}{}_{\varepsilon}) \\ +\mu_{\beta\gamma}\overset{b}{H}{}^{\alpha}+\mu^{\alpha}{}_{\beta}\overset{b}{H}_{\gamma}+\mu^{\alpha}{}_{\gamma}\overset{b}{H}_{\beta}. \tag{7.7}$$

Consequently we can state

Corollary 7.1.1. *The induced Cartan connection is also the RTM-connection on M_{n-1} determined by* (7.4) *and the h-connection $\overset{c}{\Gamma}{}^{\alpha}_{\beta\gamma}$ is also expressible in* (7.6). *In this case, expressions* (6.19) *and* (7.6) *are changeable to each other by means of* (7.7).

Note 7.1. If the *hv*-torsion tensor $\widetilde{\tau}^{\alpha}_{\beta\sigma}$ satisfying $\widetilde{\tau}^{0}_{0\gamma}=0$ is given, then the induced *RTM*-connection is determined by $\widetilde{\tau}^{\alpha}_{\beta\gamma}$ alone according to the method of Corollary 2.1.2.

Note 7.2. As well as the induced *RTM*-connection, we can consider the induced *RWTM*-connection and it becomes also the *RWTM*-connection on M_{n-1} determined by the *hv*-torsion tensor $\widetilde{\tau}^{\alpha}_{\beta\gamma}$ (or tensors $\widetilde{T}^{\alpha}{}_{\gamma}$ and $\widetilde{Z}_{\alpha\beta\gamma}$) according to Theorem 2.2 (or Corollary 2.2.1).

The connection on M_{n-1} induced from an *STM*-connection on M_n is called the *induced STM-connection.* Then we can prove that this connection satisfies all the axioms **(TM1)**~**(TM6)**, while it does not satisfy **(STM6)** because of (6. 9). In other words, the induced *STM*-connection is a *TM*-connection, but not in general an *STM*-connection. Consequently from Theorem 1. 1 we can state

Theorem 7. 2. *The induced STM-connection is the TM-connection on* M_{n-1} *determined by the tensors* $\widetilde{T}^{\alpha}{}_{\gamma}$ *and* $\widetilde{Q}_{\beta}{}^{\alpha}{}_{\gamma}$ *in expressions* (6. 6) *and* (6. 8) *respectively, and the h-torsion tensor* $\widetilde{\tau}_{\beta}{}^{\alpha}{}_{\gamma}$ *is given by*

(7. 8) $$\widetilde{\tau}_{\beta}{}^{\alpha}{}_{\gamma} = \mu^{\alpha}{}_{\beta} H_{\gamma} - \mu^{\alpha}{}_{\gamma} H_{\beta} .$$

If $\widetilde{\tau}_{\beta}{}^{\alpha}{}_{\gamma} = 0$, then from (7. 8) we have

(7. 9) $$\mu_{\alpha\beta} H_{\gamma} = \mu_{\alpha\gamma} H_{\beta} ,$$

which implies that $H_{\beta} = 0$ or there exists a positively homogeneous scalar λ of degree -3 in y^{α} such that

(7. 10) $$\mu_{\alpha\beta} = \lambda H_{\alpha} H_{\beta} \ (H_{\alpha} \neq 0).$$

If $\lambda \neq 0$, then contraction of (7. 10) by y^{α} yields $H_0 = 0$. Therefore from (5. 25) we have $\overset{b}{H}_{\alpha} = 0$. The condition $H_{\alpha} = 0$ implies also $\overset{b}{H}_{\alpha} = 0$. Differentiating this result by y^{β}, we have $\overset{b}{H}_{\alpha\beta} = N_i (B^{i}_{\alpha\beta} + G_j{}^{i}{}_{k} B^{jk}_{\alpha\beta}) = 0$. Similarly $\overset{b}{H}_{\alpha\beta\|\gamma} = N_i G_j{}^{i}{}_{k\|h} B^{jkh}_{\alpha\beta\gamma} = 0$. And so on.

Hence we can state

Corollary 7. 2. 1. *The induced STM-connection is an STM-connection on* M_{n-1} *if and only if the normal curvature vector* H_{α} *vanishes or the tensor* $\mu_{\alpha\beta}$ *is expressible in* (7. 10). *In this case, the following facts hold if* $\mu_{\alpha\beta} \neq 0$:

(7. 11) $$H_0 = 0,\ \overset{b}{H}_{\alpha} = 0,\ \overset{b}{H}_{\alpha\beta} = 0,\ N_i G_j{}^{i}{}_{kh} B^{jkh}_{\alpha\beta\gamma} = 0 \text{ and so on,}$$

where $G_j{}^{i}{}_{kh} = G_j{}^{i}{}_{k\|h}$.

The connection on M_{n-1} induced from a *GT*-connection on M_n is called the *induced GT-connection.* For a *GT*-connection, we have $Q_j{}^{i}{}_{k} = 0$. Therefore from (6. 8) we obtain

(7. 12) $$\widetilde{Q}_{\beta}{}^{\alpha}{}_{\gamma} = -\mu^{\alpha}{}_{\beta} H_{\gamma} .$$

The induced GT-connection satisfies all the axioms **(TM1)～(TM6)**. However the expression (7. 12) shows that the axiom **(GT5)** does not, in general, hold.

In other words, the induced GT-connection is a TM-connection on M_{n-1}, but not in general a GT-connection. Consequently we can state

Theorem 7. 3. *The induced GT-connection is the TM-connection on* M_{n-1} *determined by tensors* $\widetilde{T}^{\alpha}{}_{\gamma}$ *and* $\widetilde{Q}_{\beta}{}^{\alpha}{}_{\gamma}$ *in expressions* (6. 6) *and* (7. 12) *respectively.*

If $\widetilde{Q}_{\beta}{}^{\alpha}{}_{\gamma}=0$, then from (7. 12) we have

$$\mu^{\alpha}{}_{\beta}=0 \text{ or } H_{\gamma}=0. \tag{7. 13}$$

Therefore it follows from (6. 6), (6. 8), (6. 16) and (7. 13) that

$$\widetilde{T}^{\alpha}{}_{\gamma}=T^{\alpha}{}_{\gamma},\ \widetilde{T}_{\beta}{}^{\alpha}{}_{\gamma}=T^{\alpha}{}_{\gamma\|\beta}=T_{\beta}{}^{\alpha}{}_{\gamma},\ \widetilde{Q}_{\beta}{}^{\alpha}{}_{\gamma}=Q_{\beta}{}^{\alpha}{}_{\gamma}=0,\ G_{\beta}{}^{\alpha}{}_{\gamma}=\overset{b}{\Gamma}{}_{\beta}{}^{\alpha}{}_{\gamma}. \tag{7. 14}$$

In consequence of (7. 14) we have

$$\Gamma_{\beta}{}^{\alpha}{}_{\gamma}=B^{\alpha}{}_{i}(B_{\beta}{}^{i}{}_{\gamma}+\Gamma_{j}{}^{i}{}_{k}B^{jk}_{\beta\gamma})=G_{\beta}{}^{\alpha}{}_{\gamma}+T_{\beta}{}^{\alpha}{}_{\gamma},\ \Gamma^{\alpha}{}_{\gamma}=G^{\alpha}{}_{\gamma}+T^{\alpha}{}_{\gamma}, \tag{7. 15}$$

which shows that the induced GT-connection is the intrinsic GT-connection determined by $T^{\alpha}{}_{\gamma}$ (the tensor on M_{n-1} induced from a tensor $T^{i}{}_{k}$ determining the original GT-connection on M_{n}). In such a case, we shall simply say that the induced connection is *intrinsic*.

Consequently we can state

Corollary 7. 3. 1. *The induced GT-connection is a GT-connection on* M_{n-1} *if and only if the induced GT-connection is intrinsic.*

The connection on M_{n-1} induced from the IS-connection on M_n is called the *induced IS-connection*. Then it is seen that this connection satisfies all the axioms **(RTM1)～(RTM5)** and the tensors $\widetilde{T}^{\alpha}{}_{\gamma}$ and $\widetilde{Q}_{\beta}{}^{\alpha}{}_{\gamma}$ are given by

$$\widetilde{T}^{\alpha}{}_{\gamma}=-\mu^{\alpha}{}_{\gamma}H_{0}+T^{\alpha}{}_{\gamma},\ \widetilde{Q}_{\beta}{}^{\alpha}{}_{\gamma}=-\mu^{\alpha}{}_{\beta}H_{\gamma}. \tag{7. 16}$$

Then we have $\widetilde{Z}_{\alpha\beta\gamma}=\widetilde{Q}_{\alpha\beta\gamma}-\widetilde{Q}_{\beta\alpha\gamma}=0$. Differentiating $\widetilde{T}^{\alpha}{}_{\gamma}$ by y^{β}, we have

$$\widetilde{T}_{\beta}{}^{\alpha}{}_{\gamma}=\widetilde{T}^{\alpha}{}_{\gamma\|\beta}=-\mu^{\alpha}{}_{\beta\gamma}H_{0}-2\mu^{\alpha}{}_{\gamma}\overset{h}{H}_{\beta}+T_{\beta}{}^{\alpha}{}_{\gamma}+2\mu^{\alpha}{}_{\beta}T^{i}{}_{k}N_{i}B^{k}{}_{\gamma},$$

which, because of (5. 27), implies

$$g_{\varepsilon\alpha}\widetilde{T}^{\alpha}_{\beta\gamma}=\widetilde{T}_{\beta\varepsilon\gamma}=-H_0(\nu_{\varepsilon\beta\gamma}-2C_{\gamma\varepsilon\sigma}\mu^{\sigma}_{\ \beta}-2C_{\beta\varepsilon\sigma}\mu^{\sigma}_{\ \gamma})-2\mu_{\varepsilon\gamma}\overset{b}{H}_{\beta} \tag{7. 17}$$
$$+T_{\beta\varepsilon\gamma}+2\mu_{\varepsilon\beta}T^{n}_{\ \gamma},\ T^{n}_{\ \gamma}=T^{i}_{\ k}N_{i}B^{k}_{\ \gamma}.$$

Since the original connection is the *IS*-connection, the following relation holds:

$$(C_{ijr}T^{r}_{\ k})B^{ijk}_{\varepsilon\beta\gamma}=-P_{\varepsilon\beta\gamma}-½(T_{\varepsilon\beta\gamma}+T_{\beta\varepsilon\gamma}). \tag{7. 18}$$

On the other hand, the following relation holds:

$$C_{\beta\varepsilon\sigma}T^{\sigma}_{\ \gamma}=C_{jir}(\delta^{r}_{\ h}-N^{r}N_{h})T^{h}_{\ k}B^{ijk}_{\beta\varepsilon\sigma}=(C_{jih}T^{h}_{\ k})B^{jik}_{\beta\varepsilon\gamma}-\mu_{\beta\varepsilon}T^{n}_{\ \gamma}. \tag{7. 19}$$

If we apply (7. 16) (with $\widetilde{Z}_{\alpha\beta\gamma}=0$) and (7. 17)～(7. 19) to (7. 3), then we have

$$\Gamma^{\alpha}_{\beta\gamma}=\Gamma^{*\alpha}_{\beta\gamma}+H_0(C^{\alpha}_{\beta\sigma}\mu^{\sigma}_{\ \gamma}+C^{\alpha}_{\gamma\sigma}\mu^{\sigma}_{\ \beta}-C^{\sigma}_{\beta\gamma}\mu^{\alpha}_{\ \sigma})+\mu_{\beta\gamma}\overset{b}{H}{}^{\alpha}-\mu^{\alpha}_{\ \gamma}\overset{b}{H}_{\beta} \tag{7. 20}$$
$$+P^{\alpha}_{\beta\gamma}+T^{\ \alpha}_{\beta\ \gamma}+\mu^{\alpha}_{\ \beta}T^{n}_{\ \gamma}=\overset{c}{\Gamma}{}^{\alpha}_{\beta\gamma}+P^{\alpha}_{\beta\gamma}+T^{\ \alpha}_{\beta\ \gamma}+\mu^{\alpha}_{\ \beta}T^{n}_{\ \gamma}.$$

Consequently from Theorem 7.1 and Corollary 7.3.1 we have

Theorem 7. 4. *The induced IS-connection is the RTM-connection on* M_{n-1} *determined by tensors* $\widetilde{T}^{\alpha}_{\ \gamma}$ *(in* (7.16)) *and* $\widetilde{Z}_{\alpha\beta\gamma}$ (=0), *and the h-connection is expressible in* (7. 20). *The induced IS-connection is the IS-connection on* M_{n-1} *if and only if the connection in consideration is intrinsic.*

The connection on M_{n-1} induced from the *RWGT*-connection on M_n is called the *induced RWGT-connection.* This connection satisfies all the axioms **(CT1)**～**(CT4)** and tensors $\widetilde{T}^{\alpha}_{\ \gamma}$ and $\widetilde{Q}^{\ \alpha}_{\beta\ \gamma}$ are given by (7. 16).

Consequently from Corollary 2.2.1 and Theorem 7.4 we have

Corollary 7. 4. 1. *The induced RWGT-connection is the RWTM-connection on* M_{n-1} *determined by tensors* $\widetilde{T}^{\alpha}_{\ \gamma}$ *(in* (7. 16)) *and* $\widetilde{Z}_{\alpha\beta\gamma}$ (=0), *and h-connection is expressible in* (7. 20), *provided that* $\widetilde{T}^{\alpha}_{\ 0}=T^{\alpha}_{\ 0}\neq 0$.

We shall call the connection on M_{n-1} induced from a Wagner connection on M_n the *induced Wagner connection* and denote it by IWΓ. This connection satisfies **(CT1)**～**(CT4)**. From (6.6), (6.9), (2.29) and (2.30) we have

$$\widetilde{T}^{\alpha}_{\ \gamma}=-\mu^{\alpha}_{\ \gamma}\overset{b}{H}_{0}+T^{\alpha}_{\ \gamma},\ T^{\alpha}_{\ \gamma}=s^{\alpha}y_{\gamma}-\delta^{\alpha}_{\ \gamma}s_{0}-\overline{L}^{2}(C^{\ \alpha}_{\gamma\ i}s^{i}), \tag{7. 21}$$

$$\widetilde{\tau}^{\ \alpha}_{\beta\ \gamma}=\mu^{\alpha}_{\ \beta}H_{\gamma}-\mu^{\alpha}_{\ \gamma}H_{\beta}+\tau^{\ \alpha}_{\beta\ \gamma},\ \tau^{\ \alpha}_{\beta\ \gamma}=\delta^{\alpha}_{\ \beta}s_{\gamma}-\delta^{\alpha}_{\ \gamma}s_{\beta}, \tag{7. 22}$$

$$\widetilde{\tau}_{\beta\alpha\gamma} = \mu_{\beta\alpha} H_{\gamma} - \mu_{\gamma\alpha} H_{\beta} + \tau_{\beta\alpha\gamma}, \quad \tau_{\beta\alpha\gamma} = \delta_{\beta\alpha} s_{\gamma} - \delta_{\gamma\alpha} s_{\beta}.$$

Making use of (2. 10) and (7. 22) we obtain

(7. 23)
$$\widetilde{T}^{\varepsilon}{}_{\beta} = \tfrac{1}{2} g^{\varepsilon\gamma} (\tau_{\beta 0\gamma} + \tau_{0\beta\gamma} + \tau_{0\gamma\beta}) - C_{\beta}{}^{\varepsilon}{}_{\sigma} \widetilde{T}^{\sigma}{}_{0} \quad (\widetilde{T}^{\sigma}{}_{0} = g^{\sigma\gamma} \tau_{00\gamma})$$
$$= s^{\varepsilon} y_{\beta} - \delta^{\varepsilon}{}_{\beta} s_{0} - \mu^{\varepsilon}{}_{\beta} H_{0} - \overline{L}^{2} C_{\beta}{}^{\varepsilon}{}_{\sigma} s^{\sigma} \quad (H_{0} = \overset{b}{H}_{0} + T^{n}{}_{0}),$$

which coincides with (7. 21). Therefore from (2. 9), (7. 22) and (7. 23) we have

(7. 24)
$$\Gamma_{\beta}{}^{\alpha}{}_{\gamma} = \Gamma^{*}{}_{\beta}{}^{\alpha}{}_{\gamma} + H^{\alpha} \mu_{\beta\gamma} - \mu^{\alpha}{}_{\gamma} H_{\beta} + g_{\beta\gamma} s_{\beta} + s_{0} C_{\beta}{}^{\alpha}{}_{\gamma} + (y^{\alpha} C_{\beta\gamma\varepsilon}$$
$$- C_{\gamma}{}^{\alpha}{}_{\varepsilon} y_{\beta} - C_{\beta}{}^{\alpha}{}_{\varepsilon} y_{\gamma}) s^{\varepsilon} + H_{0} (C_{\gamma}{}^{\alpha}{}_{\varepsilon} \mu^{\varepsilon}{}_{\beta} + C_{\beta}{}^{\alpha}{}_{\varepsilon} \mu^{\varepsilon}{}_{\gamma} - C_{\beta}{}^{\varepsilon}{}_{\gamma} \mu^{\alpha}{}_{\varepsilon})$$
$$+ \overline{L}^{2} (C_{\gamma}{}^{\alpha}{}_{\sigma} C_{\beta}{}^{\sigma}{}_{\varepsilon} + C_{\beta}{}^{\alpha}{}_{\sigma} C_{\gamma}{}^{\sigma}{}_{\varepsilon} - C_{\sigma}{}^{\alpha}{}_{\varepsilon} C_{\beta}{}^{\sigma}{}_{\gamma}) s^{\varepsilon} \quad (H^{\alpha} = g^{\alpha\varepsilon} H_{\varepsilon}),$$

which implies

(7. 25)
$$\Gamma^{\alpha}{}_{\gamma} = G^{\alpha}{}_{\gamma} - \mu^{\alpha}{}_{\gamma} H_{0} + y_{\gamma} s^{\alpha} - \delta^{\alpha}{}_{\gamma} s_{0} - \overline{L}^{2} C_{\gamma}{}^{\alpha}{}_{\varepsilon} s^{\varepsilon}.$$

Thus we can state

Theorem 7. 5. *The induced Wagner connection* $IW\Gamma$ *is the* $RWTM$*-connection on* M_{n-1} *determined by the h-torsion tensor* $\widetilde{\tau}_{\beta}{}^{\alpha}{}_{\gamma}$ *in* (7. 22), *and the h-connection and non-linear connection are expressible in* (7. 24) *and* (7. 25) *respectively.*

The connection on M_{n-1} induced from an AMR-connection on M_{n} is called the *induced* AMR*-connection.* This connection satisfies all the axioms (**RTM1**)～(**RTM5**) and the tensors $\widetilde{T}^{\alpha}{}_{\gamma}$ and $\widetilde{\tau}_{\beta}{}^{\alpha}{}_{\gamma}$ are given as follows:

(7. 26)
$$\widetilde{T}^{\alpha}{}_{\gamma} = -\mu^{\alpha}{}_{\gamma} H_{0} + T^{\alpha}{}_{\gamma}, \quad T^{\alpha}{}_{\gamma} = \overline{f}\overline{L} h^{\alpha}{}_{\gamma} \quad (H_{0} = \overset{b}{H}_{0}),$$

where $\overline{f}(u^{\alpha}, y^{\alpha}) = f(x^{i}(u^{\alpha}), B^{i}{}_{\alpha} y^{\alpha})$ and $h^{\alpha}{}_{\gamma} = \delta^{\alpha}{}_{\gamma} - l^{\alpha} l_{\gamma}$.

(7. 27)
$$\widetilde{\tau}_{\beta}{}^{\alpha}{}_{\gamma} = \mu^{\alpha}{}_{\beta} H_{\gamma} - \mu^{\alpha}{}_{\gamma} H_{\beta} + \tau_{\beta}{}^{\alpha}{}_{\gamma}, \quad \tau_{\beta}{}^{\alpha}{}_{\gamma} = \overline{f}(l_{\beta} \delta^{\alpha}{}_{\gamma} - l_{\gamma} \delta^{\alpha}{}_{\beta}),$$
$$\widetilde{\tau}_{\beta\alpha\gamma} = \mu_{\beta\alpha} H_{\gamma} - \mu_{\gamma\alpha} H_{\beta} + \tau_{\beta\alpha\gamma}, \quad \tau_{\beta\alpha\gamma} = \overline{f}(l_{\beta} g_{\gamma\alpha} - l_{\gamma} g_{\beta\alpha}).$$

From (2. 33) we have

(7. 28)
$$s_{\beta} = -\overline{f} y_{\beta} / \overline{L} \quad \text{or} \quad s^{\beta} = -\overline{f} y^{\beta} / \overline{L}.$$

Applying (7. 28) to (7. 23), we have

(7. 29) $$\widetilde{T}^{\varepsilon}_{\beta}=-\mu^{\varepsilon}_{\beta}H_0+\bar{f}\bar{L}h^{\varepsilon}_{\beta}\quad (H_0=\overset{b}{H}_0),$$

which coincides with (7. 26). Further if we apply (7. 28) to (7. 24) and (7. 25) we obtain

(7. 30) $$\Gamma^{\alpha}_{\gamma}=G^{\alpha}_{\gamma}+\widetilde{T}^{\alpha}_{\gamma},\quad \widetilde{T}^{\alpha}_{\gamma}=-\mu^{\alpha}_{\gamma}H_0+\bar{f}\bar{L}h^{\alpha}_{\gamma},$$

(7. 31) $$\begin{aligned}\Gamma^{\alpha}_{\beta\gamma}&=\Gamma^{*\alpha}_{\beta\gamma}+H^{\alpha}\mu_{\beta\gamma}-\mu^{\alpha}_{\gamma}H_{\beta}+\bar{f}(l_{\beta}h^{\alpha}_{\gamma}-l^{\alpha}h_{\beta\gamma}-\bar{L}C^{\alpha}_{\beta\gamma})\\&\quad+H_0(C^{\alpha}_{\gamma\varepsilon}\mu^{\varepsilon}_{\beta}+C^{\alpha}_{\beta\varepsilon}\mu^{\varepsilon}_{\gamma}-C^{\varepsilon}_{\beta\gamma}\mu^{\alpha}_{\varepsilon}).\end{aligned}$$

Since $H_{\beta}=\overset{b}{H}_{\beta}+T^{n}_{\beta}=\overset{b}{H}_{\beta}+fLh^{i}_{j}N_{i}B^{j}_{\beta}=\overset{b}{H}_{\beta}$, (7. 31) is, because of (7. 6), expressible in

(7. 32) $$\Gamma^{\alpha}_{\beta\gamma}=\overset{c}{\Gamma}{}^{\alpha}_{\beta\gamma}+\bar{f}(l_{\beta}h^{\alpha}_{\gamma}-l^{\alpha}h_{\beta\gamma}-\bar{L}C^{\alpha}_{\beta\gamma}).$$

Consequently we can state

Theorem 7. 6. *The induced AMR-connection is the RTM-connection on M_{n-1} determined by the h-torsion tensor $\widetilde{\tau}^{\alpha}_{\beta\gamma}$ in* (7. 27), *and the non-linear connection and h-connection are expressible in* (7. 30) *and* (7. 31) (or (7. 32)) *respectively.*

§8. Special hypersurfaces. In this section, we shall consider various special hypersurfaces. Let us first consider a curve C: $u^{\alpha}=u^{\alpha}(s)$ (s:arc-length) in M_{n-1}. Since $x^{i}=x^{i}(u^{\alpha}(s))$ along C, the unit tangent vector is given by

(8. 1) $$l^{i}\ (=dx^{i}/ds=B^{i}_{\beta}(du^{\beta}/ds))=B^{i}_{\beta}l^{\beta}.$$

Then the absolute differentials of both sides in (8. 1) are, because of (5. 16) and (5. 20), as follows:

(8. 2) $$\begin{aligned}Dl^{i}&=(H_{\beta\gamma}l^{\beta})N^{i}du^{\gamma}+B^{i}_{\beta}Dl^{\beta}=H_{\gamma}N^{i}du^{\gamma}+B^{i}_{\beta}Dl^{\beta},\\\overset{0}{D}l^{i}&=(\overset{0}{H}_{\beta\gamma}l^{\beta})N^{i}du^{\gamma}+B^{i}_{\beta}Dl^{\beta}=H_{\gamma}N^{i}du^{\gamma}+B^{i}_{\beta}Dl^{\beta}.\end{aligned}$$

Therefore from (8. 2) we have along C

(8. 3) $$Dl^{i}/ds\ (\text{or}\ \overset{0}{D}l^{i}/ds)=(H_{\gamma}du^{\gamma}/ds)N^{i}+(Dl^{\beta}/ds)B^{i}_{\beta}.$$

Paths with respect to $ITM\Gamma$ (or $ITM\Gamma(0)$) are geodesics of M_{n-1} because of (**TM6**) (or (**T4**)). Similarly with respect to $TM\Gamma$ (or $TM\Gamma(0)$) the same fact holds on M_n. Hence it follows from (8. 3) that geodesics of M_{n-1} are those of M_n if and only if $H_\gamma = 0$. Consequently we can state

Theorem 8. 1. *A hypersurface M_{n-1} of M_n is totally geodesic with respect to $ITM\Gamma$ (or $ITM\Gamma(0)$) if and only if the normal curvature vector H_a identically vanishes.*

If $H_a = 0$ then we can deduce $\overset{b}{H}_{a\beta} = 0$. In this case, we have

$$H_{a\beta} = \overset{0}{H}_{a\beta} = T^{\;n}_{a\;\beta} + Q^{\;n}_{a\;\beta},\quad T^{\;n}_{a\;\beta} = N_i\, T^{\;i}_{j\;k}\, B^{jk}_{a\beta}.$$

Therefore taking account of Lemma 1, we obtain

$$\overset{b}{H}_{a\beta} = T^{\;n}_{a\;\beta} = 0. \tag{8. 4}$$

Conversely if (8. 4) holds then we have $H_{a\beta} = Q^{\;n}_{a\;\beta} + \mu_a H_\beta$ (or $\overset{0}{H}_{a\beta} = Q^{\;n}_{a\;\beta}$), contraction of which by y^a yields $H_\beta = 0$. Hence from Lemma 1 and Theorem 8. 1 we can state

Corollary 8. 1. 1. *Let M_{n-1} be a hypersurface of M_n endowed with the induced TM-connection $\bar{\Gamma}$ (or TM(0)-connection $\bar{\Gamma}_0$). Then the following propositions are mutually equivalent:*

(1) *M_{n-1} is totally geodesic with respect to $\bar{\Gamma}$ (or $\bar{\Gamma}_0$).*

(2) *The normal curvature vector H_a identically vanishes.*

(3) *The second fundamental tensor $H_{a\beta}$ (or $\overset{0}{H}_{a\beta}$) is expressible in* (5. 17)$_1$.

(4) *A condition* (8. 4) *always holds.*

Note 8. 1. For the $IB\Gamma$ and $IR\Gamma$, the above propositions (3) and (4) are the same.

A hypersurface M_{n-1} is called a *hyperplane of the first kind* if each path of M_{n-1} with respect to $\bar{\Gamma}$ (or $\bar{\Gamma}_0$) is a path of M_n with respect to Γ (or Γ_0).

For the induced WTM (or $WTM(0)$)-connection, the equation (8. 3) also holds. Hence we can state

Theorem 8. 2. *A hypersurface M_{n-1} of M_n is a hyperplane of the first kind with respect to the induced WTM (or WTM(0))-connection if and only if the normal curvature vector H_a identically vanishes. In this case, Lemma 1 still holds.*

The $IWΓ$ is a WTM-connection on M_{n-1} and $\overset{w}{H}_\alpha = \overset{b}{H}_\alpha + \overset{w}{T}{}^n{}_\alpha$. Hence from (2. 30) and Theorem 8. 2 we can state

Corollary 8. 2. 1. *A hypersurface M_{n-1} of M_n is a hyperplane of the first kind with respect to the induced Wagner connection if and only if the following equation holds:*

$$\overset{b}{H}_\alpha = N_i\,(\overline{L}^2\, C^i_{j\alpha}\, s^j - s^i\, y_\alpha). \tag{8. 5}$$

We shall call a curve $C: x^i = x^i(s)$ *in* M_n an *h-path with respect to* $Γ$ (*or* $Γ_0$) if the following equations hold along C:

$$Dy^i/ds = 0,\quad D(dx^i/ds)/ds = d^2x^i/ds^2 + Γ^i_{jk}(x,y)\,(dx^j/ds)\,(dx^k/ds) = 0. \tag{8. 6}$$

A hypersurface M_{n-1} is called a *hyperplane of the second kind* if any h-path in M_{n-1} with respect to $\overline{Γ}$ (or $\overline{Γ}_0$) is an h-path in M_n with respect to $Γ$ (or $Γ_0$).

Along a curve $C: x^i = x^i(u^\alpha(s))$ in M_{n-1}, from (5.1)~(5.4) we have

$$Dy^i/ds = (Dy^\alpha/ds)\,B^i_\alpha + H_\alpha\,(du^\alpha/ds)\,N^i, \tag{8. 7}$$

$$\begin{aligned} D(dx^i/ds)/ds &= (D(dy^\alpha/ds)/ds)B^i_\alpha + \{H_{\alpha\beta}(u^\varepsilon, y^\varepsilon)\,(du^\alpha/ds)\,(du^\beta/ds) \\ &\quad + \mu_{\alpha\beta}(u^\varepsilon, y^\varepsilon)\,(du^\alpha/ds)\,(Dy^\beta/ds)\}\,N^i, \\ \overset{0}{D}(dx^i/ds)/ds &= (\overset{0}{D}(dy^\alpha/ds)/ds)B^i_\alpha + \{\overset{0}{H}_{\alpha\beta}(u^\varepsilon, y^\varepsilon)(du^\alpha/ds)\,(du^\beta/ds) \\ &\quad + \mu_{\alpha\beta}(u^\varepsilon, y^\varepsilon)(du^\alpha/ds)(Dy^\beta/ds)\}\,N^i. \end{aligned} \tag{8. 8}$$

Then it follows from (8. 7) and (8. 8) that any h-path in M_{n-1} with respect to $\overline{Γ}$ (or $\overline{Γ}_0$) is an h-path in M_n with respect to $Γ$ (or $Γ_0$) if and only if $H_\alpha = 0$ and $(H_{\alpha\beta} + H_{\beta\alpha})/2 = 0$ (or $(\overset{0}{H}_{\alpha\beta} + \overset{0}{H}_{\beta\alpha})/2 = 0$).

Consequently from Corollary 8. 1. 1 we state

Theorem 8. 3. *A hypersurface M_{n-1} of M_n is a hyperplane of the second kind with respect to $ITMΓ$ (or $ITMΓ(0)$) if and only if M_{n-1} is totally geodesic and the tensor $Q_\alpha{}^n{}_\beta$ is skew-symmetric.*

Corollary 8. 3. 1. *With respect to the induced GT-connection or $ITMΓ(0)$ with $Q^\alpha_{\beta\gamma} = 0$, the following propositions are mutually equivalent:*

(1) *M_{n-1} is totally geodesic.*

(2) *M_{n-1} is a hyperplane of the first kind.*

(3) *M is a hyperplane of the second kind.*

Note 8.2. The $IH\Gamma$, $IB\Gamma$ and the induced IS-connection are practical examples for Corollary 8.3.1.

Corollary 8.3.2. *With respect to $ITM\Gamma$ (or $ITM\Gamma(0)$) with the symmetric tensor $Q_{\beta}{}^{\alpha}{}_{\gamma}$, M_{n-1} is a hyperplane of the second kind if and only if M_{n-1} is totally geodesic and the tensor $Q_{\alpha}{}^{n}{}_{\beta}$ vanishes,* i.e., $H_{\alpha\beta}=0$.

For an AMR-connection we have $Q_{j}{}^{i}{}_{k}=-Lf_{\|j}h^{i}{}_{k}+fl_{k}h^{i}{}_{j}-fLC_{j}{}^{i}{}_{k}-P_{j}{}^{i}{}_{k}$ [9], which implies $Q_{\alpha}{}^{n}{}_{\beta}=-(fLC_{j}{}^{i}{}_{k}+P_{j}{}^{i}{}_{k})N_{i}B^{jk}_{\beta\gamma}$, namely $Q_{\alpha}{}^{n}{}_{\beta}=Q_{\beta}{}^{n}{}_{\alpha}$.

Note 8.3. The $IC\Gamma$, $IR\Gamma$ and the induced AMR-connection are practical examples for Corollary 8.3.2.

Similarly from Theorem 8.2 we can state

Theorem 8.4. *A hypersurface M_{n-1} of M_{n} is a hyperplane of the second kind respect to the induced WTM (or $WTM(0)$)-connection if and only if M_{n-1} is a hyperplane of the first kind and the tensor $Q_{\alpha}{}^{n}{}_{\beta}$ is skew-symmetric.*

For the $RWGT$-connection, we have $Q_{j}{}^{i}{}_{k}=0$ and hence $Q_{\alpha}{}^{n}{}_{\beta}=0$. Consequently we have

Corollary 8.4.1. *With respect to the induced $RWGT$-connection, a hypersurface M_{n-1} of M_{n} is a hyperplane of the first kind if and only if M_{n-1} is a hyperplane of the second kind.*

For a Wagner connection, from (2.29) we have $T_{\alpha}{}^{n}{}_{\beta}-T_{\beta}{}^{n}{}_{\alpha}+Q_{\alpha}{}^{n}{}_{\beta}-Q_{\beta}{}^{n}{}_{\alpha}=\tau_{\alpha}{}^{n}{}_{\beta}=0$.

Therefore if the tensor $Q_{\alpha}{}^{n}{}_{\beta}$ is skew-symmetric, then from the above relation we obtain

$$Q_{\alpha}{}^{n}{}_{\beta}=\tfrac{1}{2}(T_{\beta}{}^{n}{}_{\alpha}-T_{\alpha}{}^{n}{}_{\beta}). \tag{8.9}$$

Since $H_{\alpha\beta}=\overset{b}{H}_{\alpha\beta}+T_{\alpha}{}^{n}{}_{\beta}+Q_{\alpha}{}^{n}{}_{\beta}=Q_{\alpha}{}^{n}{}_{\beta}$ (if $H_{\alpha}=0$), it follows that the tensor $T_{\alpha}{}^{n}{}_{\beta}$ is symmetric. Hence by virtue of (8.10) we have $Q_{\alpha}{}^{n}{}_{\beta}=0$.

Consequently we can state

Corollary 8.4.2. *With respect to the induced Wagner connection, M_{n-1} is a hyperplane of the second kind if and only if M_{n-1} is a hyperplane of the first kind and the tensor $Q_{\alpha}{}^{n}{}_{\beta}$ vanishes, that is, $H_{\alpha}=0$ and $H_{\alpha\beta}=0$.*

The projection factors B^i_α are independent of y^a, while the reciprocal projection factors B^a_i are, in general, dependent on y^a. If the B^a_i are independent of y^a, namely $B^a_{i\|\beta}=0$, then from (4. 14) we have

(8. 10) $$\mu_{\alpha\beta}=C_{ijk}B^{ij}_{\alpha\beta}N^k=0.$$

We shall say that a hypersurface M_{n-1} is *projection-factor-direction-free* or simply *pfd-free* if the equation (8. 10) holds for the M_{n-1}.

When a hypersurface $M_{n-1}: x^i=x^i(u^a)$ of M_n is given, we first choose a hypersurface element (B^i_a, N^i) and then discuss M_{n-1} at the (B^i_a, N^i). However, there are many ways to choose (B^i_a, N^i). For this, it is known [3] that a hypersurface M_{n-1} is pfd-free at any hypersurface element if and only if the enveloping space M_n is a Riemannian space.

Now we shall seek for conditions that the induced *TM* (or *TM*(0))-connection is intrinsic. A *TM* (or *TM*(0))-connection is the connection determined by tensors T^i_k and Q^i_{jk} satisfying conditions (a) and (b) respectively as follows:

$$\Gamma^i_k=G^i_k+T^i_k,\ \Gamma^i_{k\|j}=G^i_{jk}+T^i_{jk},\ \Gamma^i_{jk}=G^i_{jk}+T^i_{jk}+Q^i_{jk}.$$

Therefore the intrinsic *TM* (or *TM*(0))-connection on M_{n-1} corresponding to the above *TMΓ* (or *TMΓ*(0)) on M_n is constructed as follows:

(8. 11) $$T^a_\gamma=T^i_kB^{ak}_{i\gamma},\ \tilde{T}^a_{\beta\gamma}=T^a_{\gamma\|\beta}=T^i_{jk}B^{ajk}_{i\beta\gamma}+2T^i_kN_iB^k_\gamma\mu^a_\beta,$$

(8. 12) $$\tilde{\Gamma}^a_\gamma=G^a_\gamma+T^a_\gamma,\ \tilde{\Gamma}^a_{\beta\gamma}=G^a_{\beta\gamma}+\tilde{T}^a_{\beta\gamma}+Q^a_{\beta\gamma},\ Q^a_{\beta\gamma}=Q^i_{jk}B^{ajk}_{i\beta\gamma},$$

where $G^a_{\beta\gamma}$ is the *h*-connection of Berwald formed by $\bar{L}(u^\varepsilon, y^\varepsilon)$ and $G^a_\gamma=y^\beta G^a_{\beta\gamma}$

By virtue of (5. 11), (5. 13), (5. 27), (6. 16), (8. 11) and (8. 12), the above $\tilde{\Gamma}^a_{\beta\gamma}$ is expressible in

(8. 13) $$\tilde{\Gamma}^a_{\beta\gamma}=(\mu^a_\gamma H_0)_{\|\beta}+\mu^a_\beta H_\gamma+\Gamma^a_{\beta\gamma}\qquad \text{(for } TM\Gamma\text{)},$$

(8. 14) $$\tilde{\Gamma}^a_{\beta\gamma}=(\mu^a_\gamma H_0)_{\|\beta}+2\mu^a_\beta H_\gamma+\overset{0}{\Gamma}{}^a_{\beta\gamma}\qquad \text{(for } TM\Gamma(0)\text{)}.$$

Then if $\tilde{\Gamma}^a_{\beta\gamma}=\Gamma^a_{\beta\gamma}$ (or $\overset{0}{\Gamma}{}^a_{\beta\gamma}$), then contraction of (8. 13) (or (8. 14)) by y^β yields $\mu^a_\gamma H_0=0$ and hence $\mu^a_\beta H_\gamma=0$. Conversely this condition implies $\widetilde{\Gamma}^{\alpha}_{\beta\gamma}=\Gamma^{\alpha}_{\beta\gamma}$ (or $\overset{0}{\Gamma}{}^{\alpha}_{\beta\gamma}$) and hence $y^{\beta}\widetilde{\Gamma}^{\alpha}_{\beta\gamma}=\Gamma^{\alpha}{}_{\gamma}$. Especially for $TM\Gamma$, from (5. 10) we have $C^{\alpha}_{\beta\gamma}=C^{i}_{jk}B^{\alpha jk}_{i\beta\gamma}$. Thus we can state

Theorem 8. 5. *The induced TM (or TM(0))-connection is intrinsic if and only if M_{n-1} is pfd-free or M_{n-1} is totally geodesic with respect to ITMΓ (or ITMΓ(0)).*

Similarly we can state

Theorem 8. 6. *The induced WTM (or WTM(0))-connection is intrinsic if and only if M_{n-1} is pfd-free or M_{n-1} is a hyperplane of the first kind with respect to the induced WTM (or WTM(0))-connection.*

A hypersurface M_{n-1} is called a *hyperplane of the third kind* if the unit normal vector N^i is parallel along any curve C in M_{n-1}.

By the use of (5. 18)~(5. 20) we can deduce

(8. 15) $$DN^i=N^i{}_{|\alpha}du^{\alpha}+N^i|_{\alpha}Dy^{\alpha},\quad \overset{0}{D}N^i=N^i{}_{:\alpha}du^{\alpha}+N^i{}_{\|\alpha}Dy^{\alpha}.$$

Taking account of (4. 8), we can deduce that

(8. 16) $$N^i{}_{|\beta}=-H_{\alpha\beta}B^{\alpha}{}_jg^{ji}+g_{jk|\beta}(N^iN^j/2-g^{ij})N^k,$$
$$N^i|_{\beta}=-\mu_{\alpha\beta}B^{\alpha}{}_jg^{ji},$$

(8. 17) $$N^i{}_{:\beta}=-\overset{0}{H}_{\alpha\beta}B^{\alpha}{}_jg^{ji}+g_{jk:\beta}(N^iN^j/2-g^{ij})N^k,$$
$$N^i{}_{\|\beta}=-2\mu^{\alpha}{}_{\beta}B^i{}_{\alpha}-\mu_{\beta}N^i.$$

Then it follows from (8. 15)~(8. 16) that N^i is parallel along any curve C in M_{n-1} if and only if the following equations hold:

(8. 18) $$H_{\alpha\beta}+g_{jk|\beta}B^j{}_{\alpha}N^k=0,\ g_{jk|\beta}N^jN^k=0,\ \mu_{\alpha\beta}=0,$$

(8. 19) $$\overset{0}{H}_{\alpha\beta}+g_{jk:\beta}B^j{}_{\alpha}N^k=0,\ g_{jk:\beta}N^jN^k=0,\ \mu^{\alpha}{}_{\beta}=0,\ \mu_{\beta}=0.$$

On the other hand, the following relations hold:

(8. 20) $$g_{ij|\beta}=g_{ij|k}B^k{}_{\beta},\ g_{ij:\beta}=g_{ij|k}B^k{}_{\beta}+2C_{ijk}N^kH_{\beta}.$$

In consequence of (8. 18)~(8. 20), we obtain

(8. 21) $$H_{\alpha\beta}+g_{ij|k}B^{ik}_{\alpha\beta}N^j=0,\ g_{ij|k}B^{ik}_{\alpha\beta}N^j=0,\ \mu_{\alpha\beta}=0,$$

$$\overset{0}{H}_{\alpha\beta}+g_{i|j|k}B^{jk}_{\alpha\beta}N^j=0,\ g_{i|j|k}N^iN^jB^k_{\beta}=0,\ \mu_{\alpha\beta}=0,\ \mu_{\beta}=0. \tag{8. 22}$$

If we contract the first equation of (8. 21) (or (8. 22)) by y^{α}, then we have $H_{\beta}=0$ and hence $H_{\alpha\beta}$ (or $\overset{0}{H}_{\alpha\beta}$)$=Q_{\alpha}{}^{n}{}_{\beta}$. Therefore equations (8. 21) and (8. 22) are expressible in

$$H_{\beta}=0,\ Q_{\alpha}{}^{n}{}_{\beta}+g_{i|j|k}B^{ik}_{\alpha\beta}N^j=0,\ g_{i|j|k}N^iN^jB^k{}_{\beta}=0,\ \mu_{\alpha\beta}=0, \tag{8. 23}$$

$$H_{\beta}=0,\ Q_{\alpha}{}^{n}{}_{\beta}+g_{i|j|k}B^{ik}_{\alpha\beta}N^j=0,,\ g_{i|j|k}N^iN^jB^k{}_{\beta}=0,\ \mu_{\alpha\beta}=0, \quad \mu_{\beta}=0. \tag{8. 24}$$

Consequently we can state

Theorem 8. 6. *A hypersurface M_{n-1} of M_n is a hyperplane of the third kind with respect to ITMΓ (or ITMΓ(0)) if and only if* (8. 23) (or (8. 24)) *holds. In this case, M_{n-1} is both pfd-free and totally geodesic.*

Immediately we have

Corollary 8. 6. 1. *With respect to the induced RTM (or RTM(0))-connection, a hypersurface M_{n-1} of M_n is a hyperplane of the third kind if and only if the following fact* (1) (*or* (2)) *holds*:

(1) *M_{n-1} is both pfd-free and totally geodesic and the tensor $Q_{\alpha}{}^{n}{}_{\beta}$ vanishes, that is, $\mu_{\alpha\beta}=0$ and $H_{\alpha\beta}=0$.*

(2) *M_{n-1} is both pfd-free and totally geodesic and the tensor $Q_{\alpha}{}^{n}{}_{\beta}$ and the vector μ_{α} both vanish, that is, $H_{\alpha\beta}=\mu_{\alpha\beta}=0$ and $\mu_{\alpha}=0$.*

Note 8. 4. The *ICΓ*, the induced *AMR*-connection and the induced *IS*-connection are practical examples for the first case of Corollary 8. 6. 1, while the IRΓ is that for the second case.

A *TM*(0)-connection is called a *GT*(0)-connection if the *hv*-torsion tensor vanishes. The connection on M_{n-1} induced from a *GT*(0)-connection on M_n is called the induced *GT*(0)-connection.

For a *GT* (or *GT*(0))-connection, we have

$$g_{i|j|k}=-T_{ijk}-T_{jik}-2(C_{ijr}T^r{}_k+P_{ijk}). \tag{8. 25}$$

We consider the induced *GT*(or *GT*(0))-connection. Suppose that $H_{\alpha}=0$. Then from (4) in Corollary 8. 1. 1 we have $T_{\alpha}{}^{n}{}_{\beta}=0$ and hence $T^{n}{}_{\beta}=0$. Therefore from (8. 25) we obtain

(8. 26)
$$g_{i|j|k} B^{ik}_{\alpha\beta} N^j = -T_{n\alpha\beta} - 2(\mu_{\alpha\gamma} T^\gamma{}_\beta + P_{n\alpha\beta}),$$
$$g_{i|j|k} N^i N^j B^k{}_\beta = -2(T_{nn\beta} + \mu_\gamma T^\gamma{}_\beta + P_{nn\beta}),$$

where the index N means contraction by N^i. Applying (8. 26) to (8. 23) (or (8. 24)), we have

(8. 27) $$H_{\alpha\beta} = 0,\ \mu_{\alpha\beta} = 0,\ T_{n\alpha\beta} + 2P_{n\alpha\beta} = 0,\ T_{nn\beta} + \mu_\gamma T^\gamma{}_\beta + P_{nn\beta} = 0,$$

(8. 28) $$H_{\alpha\beta} = \mu_{\alpha\beta} = 0,\ \mu_\alpha = 0,\ T_{n\alpha\beta} + 2P_{n\alpha\beta} = 0,\ T_{nn\beta} + P_{nn\beta} = 0.$$

Thus we can state

Corollary 8. 6. 2. *With respect to the induced GT (or $GT(0)$)-connection, a hypersurface M_{n-1} of M_n is a hyperplane of the third kind if and only if* (8. 27) (or (8. 28)) *holds.*

Note 8.5. The $IH\Gamma$ and $IB\Gamma$ are practical examples for Corollary 8.6. 2 and the last two common conditions for both are given by $P_{n\alpha\beta} = 0$ and $P_{nn\beta} = 0$.

Similarly we can state

Theorem 8. 7. *With respect to the induced WTM (or $WTM(0)$)-connection, a hypersurface M_{n-1} of Mn is a hyperplane of the third kind if and only if* (8. 23) (*or* (8. 24)) *holds. In this case, M_{n-1} is pfd-free and a hyperplane of the first kind.*

Corollary 8. 7. 1. *With respect to the induced $WRTM$-connection, a hypersurface M_{n-1} of M_n is a hyperplane of the third kind if and only if M_{n-1} is a pfd-free hyperplane of the first kind together with $Q^n_{\alpha\beta} = 0$, that is, $H_{\alpha\beta} = \mu_{\alpha\beta} = 0$.*

Note 8.6. The induced Wagner connection $IW\Gamma$ is the practical example for Corollary 8.7.1.

REFERENCES

[1] M. Matsumoto: Metrical differential geometry, *Kiso Sugaku Sensho* **14**, *Shōkabo, Tokyo,* (1975).

[2] M. Matsumoto: Foundations of Finsler geometry and special Finsler spaces, (1977) (unpublished).

[3] M. Matsumoto: The induced and intrinsic Finsler connections of hypersurfaces and Finslerien projective geometry, *J. Math. Kyoto Univ.* (to appear).

[4] H. Hashiguchi: On determinations of Finsler connections by deflection tensor fields, *Rep. Sci. Kagoshima Univ.* **2** (1969), 29–39.

[5] H. Hashiguchi: On Wagner's generalized Berwald spaces, *J. Korean Math. Soc.* **12** (1 975), 51–61.

[6] H. Rund: The differential geometry of Finsler spaces, *Grundlehr. Math, Wiss.*, **101**, *Springer Verlag*, (1959).

[7] H. Yasuda: On connections of a Finsler space, *Ann. Rep. Asahikawa Med. Coll.*, **4** (1982), 1–11.

[8] H. Yasuda: On TM-connections of SK-type and G-Landsberg spaces, *Tensor, N. S.*, **40** (1983), 75–82.

[9] H. Yasuda: On TMA-connections of a Finsler space, *Tensor, N.S.*, **40** (1983), 151–158.

[10] H. Yasuda: On IS-spaces in a Finsler space, *Ann. Rep. Asahikawa Med. Coll.*, **5** (1984), 1–19.

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