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On Connections of a Finsler Space

Hiroshi YASUDA

§1. Introduction. The connection theories of a Finsler space M have been studied by many authors. These theories may be divided broadly into two types. One is the theory standing upon the viewpoint that M is constructed of line-elements and most of authors are concerned with this type of theory (see [5][1) ~ [9]). The other is the theory derived from the standpoint of tangent Minkowski spaces (for example, [1], [2], [3], [4], [10]). It is often said that the latter is more natural geometrically, but little progress has been made practically since 1962 [4]. In view of this fact, the latter theory should be given more attention.

The purpose of the present paper is to open up some possibilities for the developement of this theory. The transformations among the indicatrices at different points of M have been introduced by the present author ([11], [12]) and developed further ([13], [14]). This transformation theory has been, in this paper, applied to the theory of non-linear connections by A. Kawaguchi [3].

§2. Transformations among indicatrices. Let M be an n-dimensional Finsler space with a fundamental function $L(x, y)$. For the present, let M be endowed with the Cartan connection $C\Gamma = (\Gamma^{*i}_{jk}, N^i_k, C^i_{jk})$ or the Berwald connection $B\Gamma = (G^i_{jk}, G^i_k, O)$, where

$$(2.1) \qquad N^i_k = G^i_k, \qquad N^i_k = y^j \Gamma^{*i}_{jk}, \qquad G^i_k = y^j G^i_{jk}.$$

We consider the tangent bundle $T(M) = \bigcup_{x\in M} T_x$ over M, T_x being the tangent space at a point x of M, and take a vector field $\tilde{X}$ on $T(M)$ defined by

$$(2.2) \qquad \tilde{X} = u^i(x)\partial/\partial x^i + (v^i(x, y) - N^i_j(x, y)u^j(x))\partial/\partial y^i$$
$$(i, j = 1, 2, \ldots, n),$$

1) Numbers in brackets refer to the references at the end of the paper.

where $u^i(x)$ and $v^i(x, y)$ are the components of two vector fields on M and $v^i(x, y)$ are homogeneous functions of degree 1 in y^i.

Let $\Phi = \{\Phi_t | t \in I_\varepsilon\}$ (I_ε : an open interval) be a local one-parameter group of transformations generated by $\tilde{X}$, and Ψ_t be the restriction of Φ_t to the tangent space T_x. Then it is known [13] that the restriction Ψ_t is a mapping of T_x onto $T_{\bar{x}}$, where $T_{\bar{x}}$ is the tangent space at the image $\bar{x}$ of x by Ψ_t.

Now we consider the indicatrix bundle $I(M) = \bigcup_{x \in M} I_x$ over M, where I_x is the indicatrix at the point x and its equation is given by

$$L(x, y) = 1 \quad \text{or} \quad L^2 = g_{ij}(x, y) y^i y^j = 1 \quad (x \,;\, \textit{fixed}). \tag{2. 3}$$

In this case, I_x is considered not only as a hypersurface of T_x, but also as an $(n\text{-}1)$-dimensional submanifold of $I(M)$. If we denote the restriction of Ψ_t to I_x by $\bar{\Psi}_t$, then it is known [14] that $\bar{\Psi}_t$ is a mapping of I_x onto $I_{\bar{x}}$ if and only if the vector $v^i(x, y)$ is indicatric, i. e. $v^i y_i = 0 (y_i = g_{ij} y^j)$. Each $\bar{\Psi}_t$ is defined as a set of solutions of the following differential equation:

$$dx^i/dt = u^i(x), \qquad dy^i/dt = v^i(x, y) - N^i_j(x, y) u^j(x). \tag{2. 4}$$

The corresponding infinitesimal transformation to $\bar{\Psi}_t$ is given by

$$\bar{\Psi}_{dt} : \begin{aligned} \bar{x}^i &= x^i + u^i(x) dt, \\ \bar{y}^i &= y^i + (v^i(x, y) - N^i_j(x, y) u^j(x))\, dt, \end{aligned} \tag{2. 5}$$

where dt is an infinitesimal constant and $L(x, y) = 1$.

Each tangent space T_x can be regarded as an n-dimensional Riemannian space with a metric tensor $g_{ij}(x, y)$ and $I(M)$ as a $(2n - 1)$ -dimensional Riemannian space with torsion. In this case, the indicatrix I_x has the same Riemannian structure by means of the induced metric and connection from those of T_x or from those of $I(M)$ [11]. Further it is known [14] that each $\bar{\Psi}_t$ is an isometric mapping if and only if the following equation holds:

$$v_i|_j + v_j|_i - 2 P_{ijk} u^k = 0 \,, \tag{2. 6}$$

where $v_i|_j = \partial v_i / \partial y^j - v_r C^r_{ij}$ and $P_{ijk} = C_{ijk|r} y^r$ (the vertical short line $|$ indicates the h-covariant differentiation of Cartan).

§3. Tangent Minkowski sapces and non-linear connections. In the previous section, we have stated that each tangent space T_x can be regarded as a Riemannian space with the tensor $g_{ij}(x, y)$. Such a space is called a tangent Riemannian space. On the other hand, if the fundamental function $L(x, y)$ defines the length of a vector y^i in T_x and the vector space T_x is regarded as a centro-affine space with the origin $x = (x^i)$ whose indicatrix is given by (2. 3), then T_x is called a tangent Minkowski space. Hereafter we shall consider connections of M standing upon this view-point.

As a special case, in (2. 5) we put

$$dx^i = u^i(x)dt, \qquad v^i = -T^i_j(x, y)u^j(x), \tag{3. 1}$$

provided that $T = (T^i_j)$ is an indicatric tensor on M with respect to the upper index i such that T^i_j are homogeneous functions of degree 1 in y^i. Then (2. 5) is expressed as

$$\bar{x}^i = x^i + dx^i, \qquad \bar{y}^i = y^i - (T^i_j + N^i_j)dx^j \tag{3. 2}$$

In this case, we can consider (3. 2) as a correspondence between indicatrices I_x and $I_{\bar{x}}$ of the tangent Minkowski spaces at any two infinitesimally near points x and $\bar{x}$ of M. It is because if we neglect higher terms of dx^i in (3. 2), then $L(x, y) = 1$ implies $L(\bar{x}, \bar{y}) = 1$. Therefore, for any vector y^i in T_x we have

$$L(x, y) = L(\bar{x}, \bar{y}), \tag{3. 3}$$

that is, the length of any vector y^i is invariant under the correspondence (3. 2). From (3. 2) we obtain

$$dy^i = -(N^i_j + T^i_j)dx^j. \tag{3. 4}$$

In consequence of (3. 3) and (3. 4), we can define the absolute differential of a vector y^i as follows:

$$Dy^i = dy^i + \Gamma^i_k(x, y)dx^k, \tag{3. 5}$$

where we put

$$\Gamma^i_k = N^i_k + T^i_k. \tag{3. 6}$$

According to A. Kawaguchi [3], Γ^i_k is given by

$$(3.7)\qquad \Gamma^i_k(x,y) = h^i_r \xi^r_{jk} y^j + y^i N^j_k y_j / L^2,$$

where ξ^r_{jk} are homogeneous functions of degree 0 in y^i (or independent of y^i), $h^i_j = \delta^i_j - y^i y_j / L^2$ and $y_j = g_{ij} y^j$. Putting $\xi^r_k = \xi^r_{jk} y^j$, from (3. 6) and (3. 7) we have

$$(3.8)\qquad T^i_k = h^i_j(\xi^j_k - N^j_k),$$

which indicates that T^i_k is indicatric with respect to the index i in accordance with our assumption. Then on substituting of (3. 8) into (3. 6), the non-linear connection Γ^i_k in [3] can be obtained. Further if we let ξ^i_k be equal to Γ^i_k in (3.7) and put $T^i_k = \Gamma^i_k - N^i_k$, then we have also $y_i T^i_k = 0$, that is, a non-linear connection of type (3. 6) can be obtained reciprocally from (3. 7).

From (3. 5) and (3. 6) we obtain

$$(3.9)\qquad dL(x,y) = l_i D y^i, \qquad l_i = \partial L / \partial y^i = y_i / L.$$

For an unit vector $l^i = y^i / L$, it follows from (3. 5) and (3. 9) that

$$(3.10)\qquad L D l^i = h^i_j D y^j, \qquad l_i D l^i = 0, \qquad h^i_j = \delta^i_j - l^i l_j.$$

Let C be any curve in M represented by

$$(3.11)\qquad C : x^i = x^i(t) \quad (t\ ;\ \textit{any parameter}).$$

If a vector y^i is displaced parallelly along the curve C, then from (3. 5) and (3. 11) we have

$$(3.12)\qquad Dy^i/dt = dy^i/dt + (N^i_k + T^i_k) dx^k/dt = 0,$$

from which and (3. 9) it follows that $dL(x(t), y(t))/dt = 0$ holds along C, that is, the length of the vector y^i is invariant under any parallel displacement. Next we consider an unit vector l^i and put $l^i = dx^i/ds$, being $ds = L(x, dx)$. Then the equation of auto-parallel curves of the connection Γ^i_k, because of (2. 1) and (3. 12), becomes

$$(3.13)\qquad \begin{aligned}(D/ds)(dx^i/ds) = d^2x^i/ds^2 &+ \Gamma^{*i}_{jk}(x,l)(dx^j/ds)(dx^k/ds) \\ &+ T^i_k(x,l)(dx^k/ds) = 0.\end{aligned}$$

On the other hand, it is known [9] that the equation of geodesic curves of M is given by

$$(3.14)\qquad d^2x^i/ds^2 + \Gamma^{*i}_{jk}(x, l)(dx^j/ds)(dx^k/ds) = 0\ .$$

Consequently comparing (3.13) with (3.14), we see that auto-parallel curves of Γ^i_k are always geodesic curves of M if and only if $T^i_j(x, l)\, l^j = 0$ holds. Thus we can state

Theorem 1.[2] *The length of any vector y^i in each tangent Minkowski space is invariant under any parallel displacement by the connection Γ^i_k in consideration. Auto-parallel cvrves of the connection Γ^i_k are always geodesic curves of M if and only if the tensor $T^i_j(x, y)$ is also indicatric with respect to the lower index j.*

In the following, we shall consider only the connection Γ^i_k satisfying Theorem 1. Differentiating Γ^i_k by y^j, from (2.1) we have

$$(3.15)\qquad \partial\Gamma^i_k/\partial y^j = G^i_{jk} + T^i_{jk}, \qquad where \qquad T^i_{jk} = \partial T^i_k/\partial y^j\ .$$

For T^i_{jk}, the following relations hold:

$$(3.16)\qquad T^i_{jk} y^j = T^i_k, \qquad T^i_{jk} y^k = -\ T^i_j, \qquad T^i_{jk} y_i = -\ T_{jk},$$

where $T_{jk} = g_{jr} T^r_k$. We shall seek for the curvature tensor R^i_{kh}. This tensor is defined by

$$(3.17)\qquad R^i_{kh} = \partial\Gamma^i_k/\partial x^h - \partial\Gamma^i_h/\partial x^k + \Gamma^j_k\partial\Gamma^i_h/\partial y^j - \Gamma^j_h\partial\Gamma^i_k/\partial y^j\ ,$$

which is, because of (3.15), reducible to

$$(3.18)\qquad \begin{aligned} R^i_{kh} = y^j K^i_{jkh} &+ (\partial T^i_k/\partial x^h - T^i_{jk}\Gamma^j_h) - (\partial T^i_h/\partial x^k - T^i_{jh}\Gamma^j_k) \\ &+ T^j_k G^i_{jh} - T^j_h G^i_{jk}, \end{aligned}$$

where K^i_{jkh} is the curvature tensor of Rund. In this case, the differential equation $Dy^i = 0$ is completely integrable if and only if the tensor R^i_{kh} vanishes.

§3. Absolute differentials of covariant tensors and connections. Let us first define the absolute differential Dy_i of a covariant vector y_i. For this purpose, we choose a connection $\Gamma = (\Gamma^i_{jk})$ of M satisfying the following relation:

2) This theorem appears as Theorem 2.2 in the paper [4].

$$y^i \Gamma^i_{jk}(x, y) = \Gamma^i_k = N^i_k + T^i_k. \tag{4. 1}$$

Then we can define Dy_i as follows:

$$Dy_i = dy_i - y_j \Gamma^j_{ik} dx^k, \tag{4. 2}$$

which is, because of (2. 1), (3. 5) and (4. 1), expressible in

$$Dy_i = g_{ij} Dy^j + \{(\Gamma^{*j}_{ik} - \Gamma^j_{ik}) y_j - T_{ik}\} dx^k. \tag{4. 3}$$

Here we require that $Dy_i = g_{ij}Dy^j$. For this, Γ^i_{jk} must satisfy

$$\Gamma^j_{ik} y_j = \Gamma^{*j}_{ik} y_j - T_{ik}. \tag{4. 4}$$

In consequence of (3. 15) and (3. 16) we can find such quantities. They are expressible in

$$\Gamma^j_{ik} = G^j_{ik} + T^j_{ik} + Q^j_{ik}, \tag{4. 5}$$

provided that $Q = (Q^i_{jk})$ is an indicatric tensor with respect to indices i and j such that Q^i_{jk} are homogeneous functions of degree 0 in y^i.

Let the connection $\Gamma = (\Gamma^i_{jk})$ be symmetric, i.e. $\Gamma^i_{jk} = \Gamma^i_{kj}$. Then from (3. 16) and (4. 5) we have

$$T^i_{jk} + Q^i_{jk} = T^i_{kj} + Q^i_{kj}, \qquad Q^i_{jk} y^k = 2T^i_j, \qquad T_{jk} = T_{kj}. \tag{4. 6}$$

Let us now define the absolute differential of the tensor g_{ij} as

$$\begin{aligned} Dg_{ij} &= dg_{ij} - (g_{rj}\Gamma^r_{ik} + g_{ir}\Gamma^r_{jk}) dx^k - (g_{rj}C^r_{ik} + g_{ir}C^r_{jk}) Dy^k \\ &= g_{ij;k} dx^k + g_{ij}|_k Dy^k, \end{aligned} \tag{4. 7}$$

where

$$g_{ij;k} = \partial g_{ij}/\partial x^k - \Gamma^r_k \partial g_{ij}/\partial y^r - g_{rj}\Gamma^r_{ik} - g_{ir}\Gamma^r_{jk}, \tag{4. 8}$$

$$g_{ij}|_k = \partial g_{ij}/\partial y^k - g_{rj}C^r_{ik} - g_{ir}C^r_{jk} \equiv 0. \tag{4. 9}$$

Then from (3. 5), (3. 16), (4. 7), (4. 8) and (4. 9) we obtain

$$\begin{aligned} &Dg_{ij} = -(T_{ijk} + T_{jik} + 2C_{ijr}T^r_k + 2P_{ijk} + Q_{ijk} + Q_{jik}) dx^k, \\ &y^j Dg_{ij} = 0, \qquad where \qquad T_{ijk} = g_{jr}T^r_{ik}, \qquad Q_{ijk} = g_{jr}Q^r_{ik}. \end{aligned} \tag{4. 10}$$

For T_{ijk} and Q_{ijk}, the following relations hold:

$$(4.11)\qquad \begin{aligned} &T_{ijk}y^i = T_{jk}, \qquad T_{ijk}y^j = -T_{ik}, \qquad T_{ijk}y^k = -T_{ji}, \\ &Q_{ijk}y^i = Q_{ijk}y^j = 0\,. \end{aligned}$$

We shall call a connection Γ defined by (4.5) a *TM-connection* of M. We shall say a TM-connection Γ to be *r-metrical* if $Dg_{ij}=0$ always holds. Then it follows from (4.10) that Γ is r-metrical if and only if the following equation holds:

$$(4.12)\qquad T_{ijk} + T_{jik} + 2C_{ijr}T^r_k + 2P_{ijk} + Q_{ijk} + Q_{jik} = 0\,.$$

Further if Γ is symmetric, then from (4.6), (4.11) and (4.12) we have $T^j_{ik}=0$ and $Q^j_{ik} = -P^j_{ik}$, that is, Γ is the Cartan connection.

Let us here give a summary of the results have been obtained hitherto. Then we can state

Theorom 2. *A TM-connection Γ of M has following properties:*

(1) For the non-linear connection Γ^i_k, Theorem 1 holds.

(2) The absolute differential of a covariant vector y_i is given by

$$Dy_i = g_{ij}Dy^j.$$

(3) Γ is the Berwald connection if and only if $T^i_k=0$ and $Q^i_{jk}=0$.

(4) If Γ is symmetric, then relations (4.6) hold.

(5) Γ is r-metrical if and only if the equation (4.12) holds.

(6) Γ is the Cartan connection if and only if Γ is symmetric and r-metrical.

We consider a general tensor on M, for example, $X^i_j(x,y)$. Then the absolute differential of X^i_j is defined by

$$(4.13)\qquad \begin{aligned} &DX^i_j(x,y) = X^i_{j;k}dx^k + X^i_j|_k Dy^k, \\ &X^i_{j;k} = \partial X^i_j/\partial x^k - \Gamma^r_k \partial X^i/\partial y^r + X^r_j\Gamma^i_{rk} - X^i_r\Gamma^r_{jk}, \\ &X^i_j|_k = \partial X^i_j/\partial y^k + X^r_jC^i_{rk} - X^i_rC^r_{jk}. \end{aligned}$$

Therefore according to M. Matsumoto [7], our connection can be expressed as $TM\Gamma = (\Gamma^i_{jk}, \Gamma^i_k, C^i_{jk})$.

Since we have now many indicatric tensors on M and know the indicatrization of any tensor on M, from them or the combinations of them we can choose a tensor T^i_k in Γ^i_k and a tensor Q^i_{jk} in (4.5) in various ways. However we have at present no criterion for the choice of them. So we take an important and simple tensor T^i_k, i.e.

(4. 14) $$T^i_k = Lh^i_k = L(\delta^i_k - l^i l_k),$$

from which it follows that

(4. 15) $$T^i_{jk} = \partial T^i_k / \partial y^j = l_j h^i_k - l_k h^i_j - l^i h_{jk}.$$

Then the curvature tensor (3. 18) becomes

(4. 16) $$R^i_{kh} = y^j K^i_{jkh} + L(l_k h^i_h - l_h h^i_k).$$

Since a tensor $h_{ij} = g_{ir} h^r_j$ is called the angular metric tensor, we shall call a connection Γ^i_k defined by (4. 14) the *AM-non-linear connection.* If $R^i_{kh} = 0$, then from (4. 16) we have $K_{ikh} = -(y_k h_{ih} - y_h h_{ik})$, where $K_{ikh} = g_{ir} y^j K^r_{jkh}$, which shows that M is of constant curvature. Hence we can state

Corollary 2. 1. *For the AM-non-linear connection, the curvature tensor R^i_{kh} is given by (4. 16). If the curvature tensor vanishes, then M is of constant curvature -1.*

We shall call a TM-connection (4. 5) defined by the AM-nonlinear connection a *TMA-connection.* Let this connection be symmetric. Then from (4. 6) and (4. 15) we have

$$Q^i_{jk} - Q^i_{kj} = T^i_{kj} - T^i_{jk} = 2(l_k h^i_j - l_j h^i_k),$$
$$Q^i_{jk} y^k = 2Lh^i_j = 2T^i_j, \qquad T_{jk} = T_{kj} = Lh_{jk},$$

from which it is seen that Γ is of form

(4. 17) $$\Gamma^i_{jk} = G^i_{jk} + l_j h^i_k + l_k h^i_j - l^i h_{jk} + \Omega^i_{jk},$$

provided that $\Omega = (\Omega^i_{jk})$ is an indicatric tensor with respect to all the indices i, j and k such that Ω^i_{jk} are homogeneous functions of degree 0 in y^i and $\Omega^i_{jk} = \Omega^i_{kj}$. Hence we obtain

Corollary 2. 2. *A symmetric TMA-connection Γ is given by (4. 17).*

Let a TMA-connection Γ be r-metrical. Then because of (4. 12) and (4. 15) we obtain

(4. 18) $$Q_{ijk} + Q_{jik} = 2(h_{ij} l_k - LC_{ijk} - P_{ijk}),$$

which implies that Γ is of form

(4. 19) $$\Gamma^i_{jk} = \Gamma^{*i}_{jk} + l_j h^i_k - l^i h_{jk} - LC^i_{jk} + \tilde{\Omega}^i_{jk},$$

provided that $\tilde{\Omega} = (\tilde{\Omega}^i_{jk})$ is an indicatric tensor with respect to the indices i and j such that $\tilde{\Omega}^i_{jk}$ are homogeneous functions of degree 0 in y^i and $\tilde{\Omega}^i_{jk} + \tilde{\Omega}^j_{ik} = 0$. Consequently we have

Corollary 2. 3. *A TMA-connection Γ is r-metrical if and only if the equation (4. 18) holds. An r-metrical TMA-connection Γ is given by (4. 19).*

§5. A connection based on an isometry. As a trial, we shall utilize the equation (2. 6) in §2 to determine tensors T^i_k and T^i_{jk}.

First we consider a special *TM*-connection Γ such that

(5. 1) $$\Gamma^i_{jk} = \partial\Gamma^i_k/\partial y^j = G^i_{jk} + T^i_{jk},$$

that is, a connection obtained by putting $Q^i_{jk} = 0$ in (4. 5).

Next it follows from (3. 1) and (4. 9) that (2. 6) is reducible to

$$(T_{ijk} + T_{jik} + 2C_{ijr}T^r_k + 2P_{ijk})\,u^k = 0\,.$$

which implies that for any u^i each $\bar{\Psi}_t$ is isometric if and only if

(5. 2) $$T_{ijk} + T_{jik} + 2C_{ijr}T^r_k + 2P_{ijk} = 0\,.$$

In this case, because of (2. 4), (3. 1) and (3. 12) we see that each indicatrix I_x as a Riemannian space is isometric under the parallel displacement along any curve C also if and only if (5. 2) holds. Further it follows from (4. 12) and (5. 2) that the connection Γ is r-metrical. Contracting (5. 2) by y^k, from (4. 11) we have

(5. 3) $$T_{ij} + T_{ji} = 0\,.$$

Since $T_{ijk} = \partial T_{ik}/\partial y^i - 2C_{jri}T^r_k$, (5. 2) is reducible to

(5. 4) $$\partial T_{jk}/\partial y^i + \partial T_{ik}/\partial y^j - 2C^r_{ij}T_{rk} + 2P_{ijk} = 0\,,$$

provided that

(5. 5) $$T_{ij}y^i = 0\,, \qquad T_{ij}y^j = 0\,.$$

Further we can eliminate the terms of partial derivatives, that is

(5. 6) $$T_{ri}C^r_{jk} + T_{rj}C^r_{ki} + T_{rk}C^r_{ij} = 3P_{ijk}\,.$$

Now we are confronted with a problem if there exists a tensor T_{ij} satisgying (5. 4) under the conditions (5. 3) and (5. 5).

Let first M be of dimension 2. Then from (5. 3) and (5. 5) we have

$$(5.\ 7)\qquad T_{ii} = 0\ , \qquad T_{12} = -\ T_{21}\ , \qquad T_{12}y^1 = T_{21}y^2 = 0\ ,$$

from which it follows that $T_{ij} = 0$ and hence $P_{ijk} = 0$. Hence we obtain

Theorem 3. *Let M be a two-dimensional Finsler space. Then if M allows an indicatric and skew-symmetric tensor T_{ij} satisfying (5. 4), then M is a Landsberg space and the connection Γ in consideration becomes the Cartan (or Berwald) connection.*

We shall call a solution of the equation (5. 4) such that $T_{ij} = 0$ and $P_{ijk} = 0$ are satisfied the *trivial solution* of (5. 4).

In a general case, i. e. $n \geqq 3$, there exists the trivial solution without fail in view of the form of (5. 4). However it is for the present open if there exists any non-trivial solution.

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(Mathematics, Asahikawa Medical College)