Ann. Rep. Asahikawa Med. Coll.
1980, Vol. 2 1 ~ 21

On the Curvature Tensors of the Indicatrix Bundle over a Finsler Space

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Introduction. Let M be an n-dimensional Finsler space with a fundamental function $F(x, y)$, and let $T(M)$ and $T^*(M)$ be the tangent bundle and cotangent bundle over M. Now we consider a mapping

$$\Psi : (x, y) \in T(M) \to (x, p) \in T^*(M), \quad \text{where} \quad p_i = \partial F / \partial y^i .$$

Then $N = \Psi(T(M))$ is a hypersurface of $T^*(M)$ and called the p-manifold of M, which was first introduced and studied by M. Kurita [4][1].

Afterwards the p-manifold N was studied concretely and combined to the theory of A. Deicke ([1], [2]) by H. Yasuda ([8], [9]). The p-manifold N is in fact the *figuratrix bundle* over M. Hereafter we shall use this terminology and l_i instead of p_i. The figuratrix bundle was applied to a study of Finsler spaces with absolute parallelism of line-elements [10].

Similarly we can consider the *indicatrix bundle* $L = \underset{x \in M}{U} I_x$ over M, I_x being the indicatrix at a point x of M, and introduce a metric on L in a natural way. In this case, L is isometric to N by a mapping : $(x^i, l^i) \in L \to (x^i, l_i) \in N$ defined by $l_i = g_{ij} l^j$, where $l^j = y^j / F(x, y)$. Therefore, we can use the methods constructed already in N for the studies of L itself and their applications.

Along the above statement, the indicatrices of M and curves in L were investigated in the papers [11] and [12] where L is endowed with the D-connection. Especially, the indicatrix bundle L endowed with the K-

1) Numbers in brackets refer to the references at the end of the paper.

connection was treated in the paper [13].

On the other hand, from a standpoint of the theory of Finsler bundle and tangent bundle the indicatrix bundle L over M is constructed systematically and studied by M. Matsumoto [6], and the metric and the connection introduced there correspond to our metric and the K—connection respectively.

In consideration of the $D-$ and $K-$connections, Landsberg spaces are treated in the paper [14].

In the present paper, we shall reconstruct the indicatrix bundle L from a standpoint of the theory of implicit functions and differential forms and investigate the curvature tensors on L when L is endowed with the K_0-, D_0-, $K-$ and $D-$connections. Especially, for the D_0- and $D-$connections we have found many new tensors on M. As for the geometrical meanings of these tensors, we have obtained the considerable results when M is a Landsberg space. However, the most part of our problem remains to be solved. The terminologies and notations are referred to the papers ([12], [13]) unless otherwise stated.

§ 1. **Construction of the indicatrix bundle.** The indicatrix bundle L in Introduction can be constructed over M globally. However, it is in fact enough to consider L at a coordinate neighborhood.

Let U be a coordinate neighborhood with coordinates (x^i) $(i=1, 2, \cdots, n)$ of M. Then, if we denote by $y=y^i\partial/\partial x^i$ an element of the tangent space T_x at a point x of M, the canonical coordinates of $\overline{U}=\bigcup_{x\in U} T_x$ are expressed in (x^i, y^i). The indicatrix bundle L is a hypersurface of $T(M)$ and its local equation is given by

$$(1.1) \qquad G=F(x, y)-1=0.$$

On L we have from (1.1)

$$(1.2) \qquad \partial G/\partial x^i=\gamma_o{}^o{}_i, \qquad \partial G/\partial y^i=l_i\ ,$$

where $\gamma_k{}^j{}_i$ are the Christoffel symbols formed by the metric tensor g_{ij} and $\gamma_o{}^o{}_i=\gamma_k{}^j{}_i l^k l_j$. Since the vector l_i is presupposed as a non—zero one, we can assume $\partial G/\partial y^n=l_n\neq 0$ without loss of generality. Therefore according to the theorem of implicit functions, there exists a neighborhoob $\widetilde{U}$ such that the equation (1.1) can be solved as

(1 . 3) $\quad y^n = y^n(x^1,\ x^2,\ \cdots,\ x^n,\ y^1,\ y^2,\ \cdots,\ y^{n-1})$

and the following relations hold good in $\widetilde{U}$ by virtue of (1 . 2):

(1 . 4) $\quad \partial y^n/\partial x^i = -\gamma_o{}^o{}_i/l_n,\ \ \partial y^n/\partial y^\lambda = -l_\lambda/l_n(\lambda=1,\ 2,\ \cdots,\ n-1).$

In this case, if we substitute (1 . 3) in (1 . 1), then the expression (1. 1) becomes an identity and hence $l^i = y^i/F = y^i$. Therefore, for (1 . 3) and (1 . 4) we can rewrite as

(1 . 3)′ $\quad l^n = l^n(x^1,\ \cdots,\ x^n,\ l^1,\ \cdots,\ l^{n-1}),$

(1 . 4)′ $\quad \partial l^n/\partial x^i = -\lambda_o{}^o{}_i/l_n,\ \ \partial l^n/\partial l^\lambda = -l_\lambda/l_n.$

Now we denote by $((\partial/\partial x^i)_L,\ (\partial/\partial l^\lambda)_L)$ the natural frame at a point $(x^i,\ l^\lambda)$ of L and consider the inclusion mapping $\iota : L \rightarrow T(M)$.

For any homogeneous function $f(x^i,\ y^i)$ of degree O in y^i, we have

(1 . 5) $\quad (\partial/\partial l^i) f = F(\partial/\partial y^i) f$ *i. e.* $\partial/\partial l^i = F\ \partial/\partial y^i.$

Then it follows from (1 . 4)′ and (1 . 5) that

$$\iota_*(\partial/\partial x^i)_L = \partial/\partial x^i - (\gamma_o{}^o{}_i/l_n) F \partial/\partial y^n,$$

(1 . 6)

$$\iota_*(\partial/\partial l^\lambda)_L = F\{\partial/\partial y^\lambda - (l_\lambda/l_n)\ \partial/\partial y^n\},$$

where $(\partial/\partial x^i,\ \partial/\partial y^i)$ is the natural frame at a point $(x^i,\ y^i)$ of $T(M)$.

Remark. At the present stage, since $F=1$ we have $\partial/\partial y^i = \partial/\partial l^i$. Especially when the vector $\partial/\partial l^i$ operates to homogeneous objects of degree O in y^i, (1 . 5) is valid and it corresponds to the third covariant differentiation $\|i$ of Cartan.

In this case, it is seen from (1 . 6) that the coframe $(\iota^*(dx^i), \iota^*(dl^\lambda))$ is dual to $((\partial/\partial x^i)_L, (\partial/\partial l^\lambda)_L)$, and from (1 . 3)′ and (1 . 4)′ that

(1 . 7) $\quad \iota^*(dl^n) = -(\gamma_o{}^o{}_i/l_n)\ \iota^*(dx^i) - (l_\lambda/l_n)\ \iota^*(dl^\lambda).$

In the sequel, we shall omit the symbols ι_* and ι^* when no confusion occurs.

Now, let us consider the Cartan connection $\Gamma^*{}_j{}^i{}_k$ on M and the non-linear connection $\Gamma^*{}_k{}^i{}_j y^k$ on $T(M)$. Then, denoting by $(\bar{e}_i)$ the horizontal lift of $(\partial/\partial x^i)$ to $T(M)$ with respect to this non-linear connection, we have

$$(1.8)\qquad \bar{e}_i = \partial/\partial x^i - \Gamma^*{}_k{}^j{}_i y^k \partial/\partial y^j .$$

Further, if we put

$$(1.9)\qquad \bar{e}_{(i)} = \bar{e}_{n+i} = \partial/\partial y^i, \quad Dy^i = dy^i + \Gamma^*{}_k{}^i{}_j y^k dx^j ,$$

then it is seen from (1.8) and (1.9) that the coframe (dx^i, Dy^i) is dual to $(\bar{e}_i, \bar{e}_{(i)})$. Then, carrying the coframe (dx^i, Dy^i) back to L by ι^*, from (1.7) and (1.9) we obtain the coframe (dx^i, Dl^i) on L and

$$(1.10)\qquad \begin{aligned} Dl^\lambda &= dl^\lambda + N^\lambda{}_j dx^j , \\ Dl^n &= (N^n{}_i - \gamma_o{}^o{}_i/l_n) dx^i - (l_\lambda/l_n) dl^\lambda , \end{aligned}$$

where we put $N^i{}_j = \Gamma^*{}_o{}^i{}_j = \Gamma^*{}_k{}^i{}_j l^k$. Moreover if we denote by $(e_i, e_{(\lambda)})_L$ the frame dual to (dx^i, Dl^λ), we have

$$(1.11)\qquad (e_i)_L = (\partial/\partial x^i)_L - N^\lambda{}_i (e_{(\lambda)})_L , \quad (e_{(\lambda)})_L = (\partial/\partial l^\lambda)_L .$$

Then it follows from (1.5), (1.6), (1.8), (1.9) and (1.11) that

$$(1.12)\qquad \begin{aligned} (e_i)_L &= \partial/\partial x^i - N^j_i \partial/\partial l^j = \bar{e}_i , \\ (e_{(\lambda)})_L &= \partial/\partial l^\lambda - (l_\lambda/l_n) \partial/\partial l^n . \end{aligned}$$

In the following, we shall write $(e_i)_L$ and $(e_{(\lambda)})_L$ as e_i and $e_{(\lambda)}$ simply.

A metric $d\sigma^2$ on L is introduced in a natural way ([2], [9]) by

$$(1.13)\qquad d\sigma^2 = g_{ij} dx^i dx^j + g_{ij} Dl^i Dl^j .$$

Now, let us find an orthonormal frame and coframe on L with respect to the above metric. For this purpose, we first choose n vector fields ζ^i_a $(a=1, 2, \cdots, n)$ on M satisfying

$$(1.14)\qquad \zeta^i_n = l^i , \quad g_{ij} \zeta^i_a \zeta^j_b = \delta_{ab} .$$

And further if we denote by (ζ^a_i) the inverse of the matrix (ζ^i_a), then from (1.14) we have

$$(1.15)\qquad g^{ij} = \sum_a \zeta^i_a \zeta^j_a, \quad g_{ij} = \sum_a \zeta^a_i \zeta^a_j, \quad l_i = \zeta^n_i, \quad \zeta^i_a = g^{ij}\zeta^a_j,$$

$$\zeta^i_\alpha l_i = \zeta^\alpha_i l^i = 0 \quad (\alpha = 1, 2, \cdots, n-1),$$

where g^{ij} is the reciprocal tensor of g_{ij}. Put

$$(1.16)\qquad e_a = \zeta^i_a e_i, \quad e_{(\alpha)} = \zeta^i_\alpha \partial/\partial l^i,$$

$$(1.17)\qquad \omega^a = \zeta^a_i dx^i, \quad \omega^{(\alpha)} = \zeta^\alpha_i Dl^i.$$

Then we can state

Proposition 1. *The frame $(e_a, e_{(\alpha)})$ and coframe $(\omega^a, \omega^{(\alpha)})$ formed by (1.16) and (1.17) are dual to each other. And they are an orthonormal frame and coframe with respect to the metric defined by (1.13).*

Proof. We know already that $(e_i, e_{(\lambda)})$ and (dx^i, Dl^λ) are dual to each other. So we have

$$(1.18)\qquad dx^i(e_j) = \delta^i_j, \quad dx^i(e_{(\lambda)}) = Dl^\lambda(e_j) = O, \quad Dl^\lambda(e_{(\mu)}) = \delta^\lambda_\mu.$$

From (1.12) we get

$$(1.19)\qquad e_{(\alpha)} = \zeta^\lambda_\alpha e_{(\lambda)}.$$

If we put $\eta^\alpha_\lambda = \zeta^\alpha_\lambda - (l_\lambda/l_n)\zeta^\alpha_n$, then we obtain

$$(1.20)\qquad \zeta^\lambda_\beta \eta^\alpha_\lambda = \delta^\alpha_\beta, \quad \omega^{(\alpha)} = \eta^\alpha_\lambda Dl^\lambda.$$

Thus on making use of (1.16)~(1.20), we have

$$(1.21)\qquad \omega^a(e_b) = \delta^a_b, \quad \omega^a(e_{(\alpha)}) = \omega^{(\alpha)}(e_a) = O, \quad \omega^{(\alpha)}(e_{(\beta)}) = \delta^\alpha_\beta.$$

Next, it follows from (1.13), (1.15) and (1.17) that

$$(1.22)\qquad \sum_a \omega^a\omega^a = g_{ij}dx^i dx^j, \quad \sum_\alpha \omega^{(\alpha)}\omega^{(\alpha)} = g_{ij}Dl^i Dl^j, \quad d\sigma^2 = \sum_a \omega^a\omega^a + \sum_\alpha \omega^{(\alpha)}\omega^{(\alpha)},$$

that is, the coframe $(\omega^a, \omega^{(\alpha)})$ is an orthonormal one with respect to the metric and so is the frame $(e_a, e_{(\alpha)})$. Q. E. D.

A frame and coframe introduced in proposition 1 are usually called an *adapted orthogonal frame* and *coframe* respectively. And it follows from (1 . 22) that the components of the metric tensor with respect to $(e_a, e_{(\alpha)})$ are given by δ_{AB} $(A, B = 1, 2, \cdots, 2n-1)$.

If we put $e_{(i)} = \zeta^\alpha_i e_{(\alpha)}$, from (1 . 16) we have

$$(1\ .\ 23)\qquad e_{(i)} = h^j_i \partial / \partial l^j, \quad \text{where } h^j_i = \delta^j_i - l^j l_i,$$

In this case, the frame $(e_i, e_{(i)})$ is considered also as a frame on L, but it should be noticed that $(e_{(i)})$ are not independent because of $l^i e_{(i)} = O$.

§ 2. **Connections and torsion tensors.** Though the choice of metrical connectons on L with respect to the metric (1 . 13) is highly arbitrary, in this paper we shall consider the four connections, that is, K_0-, $K-$, D_0- $D-$connections. Hereafter we take an adapted orthogonal coframe $(\omega^a, \omega^{(\alpha)})$.

Let ω^A_B be the connection forms with respect to $(\omega^a, \omega^{(\alpha)})$ of a connection Γ on L. Since $D\delta_{AB} = -\omega^B_A - \omega^A_B$, any metrical connection Γ is given by

$$(2\ .\ 1)\qquad \Gamma = (\omega^A_B), \quad \omega^A_B = -\omega^B_A.$$

Firstly, the K_0-connection is defined as follows: In (2 . 1),

$$(2\ .\ 2)\qquad \begin{aligned} &\omega^{(\alpha)}_{(\beta)} = \omega^\alpha_\beta, \quad \omega^\alpha_n = \omega^{\ a}_{(\beta)} = \omega^n_b = \omega^{(\beta)}_b = O, \quad \omega^\alpha_\beta = \Gamma^\alpha_{\beta c}\omega^c + \Gamma^\alpha_{\beta(\gamma)}\omega^{(\gamma)}, \\ &\Gamma^\alpha_{\beta c} = -\zeta^\alpha_{i|j}\zeta^i_\beta\zeta^j_c, \quad \Gamma^\alpha_{\beta(\gamma)} = -\zeta^\alpha_i|_j\zeta^i_\beta\zeta^j_\gamma. \end{aligned}$$

Secondly, the $K-$connection is defined as follows: In (2 . 1),

$$(2\ .\ 3)\qquad \begin{aligned} &\omega^a_b = \Gamma^a_{bc}\omega^c + \Gamma^a_{b(\gamma)}\omega^{(\gamma)}, \quad \omega^{(\alpha)}_{(\beta)} = \omega^\alpha_\beta, \quad \omega^{\ a}_{(\beta)} = \omega^{(\beta)}_b = O, \\ &\Gamma^a_{bc} = -\zeta^a_{i|j}\zeta^i_b\zeta^j_c, \quad \Gamma^a_{b(\gamma)} = -\zeta^a_i|_j\zeta^i_b\zeta^j_\gamma. \end{aligned}$$

Thirdly, the $D-$connection is defined as follows: In (2 . 1), ω^a_b and $\omega^{(\alpha)}_{(\beta)}$ are the same as in (2 . 3), $-\omega^{(\alpha)}_b = \omega^{\ b}_{(\alpha)} = \Gamma^{(\alpha)}_{b\ c}\omega^c + \Gamma^{(\alpha)}_{b(\gamma)}\omega^{(\gamma)}$ and

$$(2\ .\ 4)\qquad \begin{aligned} &\Gamma^{(\alpha)}_{b\ c} = B^i{}_{jk}\zeta^\alpha_i\zeta^j_b\zeta^k_c, \quad \Gamma^{(\alpha)}_{b(\gamma)} = P^i{}_{jk}\zeta^\alpha_i\zeta^j_b\zeta^k_\gamma, \\ &B^i{}_{jk} = A^i{}_{jk} + R^i{}_{jk}, \quad P^i{}_{jk} = P^i_{o\ jk} = A^i{}_{jk|o}, \quad R^i{}_{jk} = R^i_{o\ jk}. \end{aligned}$$

Lastly, the D_o–connection is defined as follows: In (2 . 1), the forms ω^α_β, $\omega^{(\alpha)}_{(\beta)}$, ω^a_n and ω^n_b are the same as in (2 . 2), while $\omega^{(\alpha)}_b$ and $\omega_{(\alpha)}^{\ b}$ are the same as in (2 . 4).

The torsion form τ^A and tensor T^A_{BC} on L are given by

$$(2.5)\qquad \tau^A = d\omega^A - \omega^B \wedge \omega^A_B = \frac{1}{2} T^A_{BC}\, \omega^B \wedge \omega^C \quad (T^A_{BC} + T^A_{CB} = O).$$

For the D–connection, from (1 . 17) and (2 . 4) we have [9]

$$(2.6)\qquad \begin{aligned} &T^a_{bc} = T_{(\beta)(\gamma)}^{\ A} = T^{(\alpha)}_{(\beta)c} = O,\quad T_{(\beta)c}^{\ a} = -T_{c(\beta)}^{\ a} = -R_{ijk}\,\zeta^i_\beta \zeta^j_a \zeta^k_c \\ &T^{(\alpha)}_{b\ c} = -T_{(\alpha)c}^{\ b} = R_{ijk}\,\zeta^i_\alpha \zeta^j_b \zeta^k_c, \quad R_{ijk} = g_{ih} R^h{}_{jk}, \end{aligned}$$

from which it follows that $T^A_{BC} = O$ if and only if $R^i{}_{jk} = O$, and that T^A_{BC} are skew–symmetric in all indices A, B and C, that is, a path in L coincides with an extremal in L [2]. Accordingly we have

Proposition 2. *The D–connection is the Riemannian one if and only if M is a space with absolute parallelism of line–elements. With respect to the D–connection, a path in L coincides with an extremal in L.*

For the K–connection, from (1 . 17) and (2 . 3) we have ([6], [9], [13])

$$(2.7)\qquad \begin{aligned} &T^a_{bc} = T_{(\beta)(\gamma)}^{\ a} = T_{(\beta)(\gamma)}^{\ (\alpha)} = O,\quad T_{(\gamma)b}^{\ a} = -T_{b(\gamma)}^{\ a} = A^i_{jk}\,\zeta^a_i \zeta^j_b \zeta^k_\gamma, \\ &T^{(\alpha)}_{bc} = -T^{(\alpha)}_{cb} = R^i{}_{jk}\,\zeta^\alpha_i \zeta^j_b \zeta^k_c,\quad T^{(\alpha)}_{b(\gamma)} = -T^{(\alpha)}_{(\gamma)b} = P^i{}_{jk}\,\zeta^\alpha_i \zeta^j_b \zeta^k_\gamma, \end{aligned}$$

For the K_o–connection, from (1 . 17) and (2 . 2) we have [9]

$$(2.8)\qquad \begin{aligned} &T^a_{bc} = T_{(\beta)(\gamma)}^{\ a} = T_{(\beta)(\gamma)}^{\ (\alpha)} = O,\quad T_{(\gamma)b}^{\ a} = -T_{b(\gamma)}^{\ a} \\ &\qquad = (A^i{}_{jk} + l_j h^i_k + l_k h^i_j)\,\zeta^a_i \zeta^j_b \zeta^k_\gamma, \\ &T^{(\alpha)}_{b\ c} = -T^{(\alpha)}_{cb} = R^i{}_{jk}\,\zeta^\alpha_i \zeta^j_b \zeta^k_c,\quad T^{(\alpha)}_{b(\gamma)} = -T^{(\alpha)}_{(\gamma)b} = P^i{}_{jk}\,\zeta^\alpha_i \zeta^j_b \zeta^k_\gamma. \end{aligned}$$

Assume $T_{b(\gamma)}^{\ a} = O$. Then from (2 . 8) we obtain $A^i{}_{jk} + l_j h^i_k + l_k h^i_j = O$, contraction of which by l^k yields $h^i_j = O$, contrary to hypothesis.

For the D_o–connection, from (1 . 17), (2 . 2) and (2 . 4) we have

$$(2.9)\qquad T^a_{bc} = T_{(\beta)(\gamma)}^{\ A} = T^{(\alpha)}_{(\beta)c} = O,\quad T^{(\alpha)}_{b\ c} = -T^{(\alpha)}_{cb} = R^i{}_{jk}\,\zeta^\alpha_i \zeta^j_b \zeta^k_c,$$

$$T_{(\beta)c}^{\;\;a} = -T_{c(\beta)}^{\;\;a} = (l_j h_{ik} + l_i h_{jk} - R_{kij})\, \zeta_a^i \zeta_c^j \zeta_\beta^k .$$

Assume $T_{BC}^A = O$. Then in the same way as before we have $h_{jk} = O$. Thus by virtue of (2 . 7) and the above proofs we can state

Proposition 3. *The K—connection is symmetric if and only if M is locally Euclidean. For either of the K_o— and D_o—connections, the torsion tensor T_{BC}^A never vanishes.*

§ 3. **Curvature tensors.** Let Ω_B^A and K_{BCD}^A be the curvature form and tensor on L. Then they are defined by ([9], [13])

$$\Omega_B^A = \omega_B^C \wedge \omega_C^A - d\omega_B^A = \frac{1}{2} K_{B\,CD}^{A}\, \omega^C \wedge \omega^D \;\; (K_{BCD}^A = -K_{BDC}^A), \tag{3 . 1}$$

which is reducible to

$$\Omega_B^A = \frac{1}{2} R_{B\,cd}^{A}\, \omega^c \wedge \omega^d + P_{Bc(\sigma)}^A\, \omega^c \wedge \omega^{(\sigma)} + \frac{1}{2} S_{B(\gamma)(\sigma)}^{A}\, \omega^{(\gamma)} \wedge \omega^{(\sigma)} . \tag{3 . 2}$$

First, for the K—connection we know ([9], [13]) that

$$\Omega_b^a = \omega_b^c \wedge \omega_c^a - d\omega_b^a = \frac{1}{2} R_{b\,cd}^{\;a}\, \omega^c \wedge \omega^d + P_{b\,c(\sigma)}^{\;a}\, \omega^c \wedge \omega^{(\sigma)} + \frac{1}{2} S_{b(\gamma)(\sigma)}^{\;a}\, \omega^{(\gamma)} \wedge \omega^{(\sigma)} ,$$

$$R_{b\,cd}^{\;a} = R_{j\,kh}^{\;i}\, \zeta_i^a \zeta_b^j \zeta_c^k \zeta_d^h ,\quad P_{b\,c(\sigma)}^{\;a} = P_{j\,kh}^{\;i}\, \zeta_i^a \zeta_b^j \zeta_\sigma^k \zeta^h , \tag{3 . 3}$$
$$S_{b(\gamma)(\sigma)}^{\;a} = S_{j\,kh}^{\;i}\, \zeta_i^a \zeta_b^j \zeta_\gamma^k \zeta_\sigma^h ,$$

where $R_{j\,kh}^{\;i}$, $P_{j\,kh}^{\;i}$ and $S_{j\,kh}^{\;i}$ are the h—, hv—, v—curvature tensors on M.

Next, for the K_o—connection we know [9] that

$$\Omega_{(\beta)}^{(\alpha)} = \Omega_\beta^\alpha = \omega_\beta^\gamma \wedge \omega_\gamma^\alpha - d\omega_\beta^\alpha = \frac{1}{2} R_{\beta\,cd}^{\;\alpha}\, \omega^c \wedge \omega^d + P_{\beta\,c(\sigma)}^{\;\alpha}\, \omega^c \wedge \omega^{(\sigma)} + \frac{1}{2} S_{\beta(\gamma)(\sigma)}^{\;\alpha}\, \omega^{(\gamma)} \wedge \omega^{(\sigma)} ,$$

$$R_{(\beta)\,cd}^{\;(\alpha)} = R_{\beta\,cd}^{\;\alpha} = R_{j\,kh}^{\;i}\, \zeta_i^\alpha \zeta_\beta^j \zeta_c^k \zeta_d^h ,$$

$$P_{(\beta)\,c(\sigma)}^{\;(\alpha)} = P_{\beta\,c(\sigma)}^{\;\alpha} = P_{j\,kh}^{\;i}\, \zeta_i^\alpha \zeta_\beta^j \zeta_c^k \zeta_\sigma^h , \tag{3 . 4}$$

$$S_{(\beta)(\gamma)(\sigma)}^{\;(\alpha)} = S_{\beta\,\gamma\sigma}^{\;\alpha} = \widetilde{S}_{j\,kh}^{\;i}\, \zeta_i^\alpha \zeta_\beta^j \zeta_\gamma^k \zeta_\sigma^h ,$$

$$\widetilde{S}_{j\,kh}^{\;i} = S_{j\,kh}^{\;i} + h_{jk} h_h^i + h_{jh} h_k^i .$$

Now, if we denote by $''R_{j\,kh}^{\;i}$ and $'P_{j\,kh}^{\;i}$ the *semi—indicatrized tensor* of $R_{j}{}^{i}{}_{kh}$ and the indicatrized tensor of $P_{j}{}^{i}{}_{kh}$, then we have

$$(3.5)\quad {}''R_{j}{}^{i}{}_{kh} = R_{j}{}^{i}{}_{kh} + R_{jkh} l^{i} - R^{i}{}_{kh} l_{j}\,,\qquad {}'P_{j}{}^{i}{}_{kh} = P_{j}{}^{i}{}_{kh} + P_{jkh} l^{i} - P^{i}{}_{kh} l_{j}\,,\quad P_{jkh} = g_{ij} P^{i}{}_{kh}\,,$$

(I)　The K_o-connection.　From (2 . 2) we have

$$(3.6)\qquad \Omega^{a}_{n} = \Omega^{n}_{b} = \Omega^{(\alpha)}_{b} = \Omega^{a}_{(\beta)} = O.$$

Then, from (3 . 2), (3 . 4), (3 . 5) and (3 . 6) we have

$$(3.7)_1\quad R^{a}_{b\,cd} = {}''R_{j}{}^{i}{}_{kh}\,\zeta^{a}_{i}\zeta^{j}_{b}\zeta^{k}_{c}\zeta^{h}_{d}\,,\quad P^{a}_{b\,c(\sigma)} = {}'P_{j}{}^{i}{}_{kh}\,\zeta^{a}_{i}\zeta^{j}_{b}\zeta^{k}_{c}\zeta^{h}_{\sigma}\,,\quad S^{a}_{b(\gamma)(\sigma)} = \widetilde{S}_{j}{}^{i}{}_{kh}\,\zeta^{a}_{i}\zeta^{j}_{b}\zeta^{k}_{\gamma}\zeta^{h}_{\sigma}\,,$$

$$(3.7)_2\quad R^{a}_{(\beta)cd} = P^{a}_{(\beta)c(\sigma)} = S_{(\beta)}{}^{a}{}_{(\gamma)(\sigma)} = O,$$

$$(3.7)_3\quad R^{(\alpha)}_{b}{}_{cd} = P^{(\alpha)}_{b}{}_{c(\sigma)} = S^{(\alpha)}_{b(\gamma)(\sigma)} = O\,,$$

$$(3.7)_4\quad R^{(\alpha)}_{(\beta)cd} = {}''R_{j}{}^{i}{}_{kh}\,\zeta^{\alpha}_{i}\zeta^{j}_{\beta}\zeta^{k}_{c}\zeta^{h}_{d}\,,\quad P^{(\alpha)}_{(\beta)c(\sigma)} = {}'P_{j}{}^{i}{}_{kh}\,\zeta^{\alpha}_{i}\zeta^{j}_{\beta}\zeta^{k}_{c}\zeta^{h}_{\sigma}\,,\quad S^{(\alpha)}_{(\beta)(\gamma)(\sigma)} = \widetilde{S}_{j}{}^{i}{}_{kh}\,\zeta^{\alpha}_{i}\zeta^{j}_{\beta}\zeta^{k}_{\gamma}\zeta^{h}_{\sigma}\,.$$

As well known, any indicatrix of M is locally flat if and only if $\widetilde{S}_{j}{}^{i}{}_{kh} = O$.　It follows from $(3.7)_1$ and $(3.7)_4$ that $K^{A}_{B\,CD} = O$ if and only if ${}''R_{j}{}^{i}{}_{kh} = {}'P_{j}{}^{i}{}_{kh} = \widetilde{S}_{j}{}^{i}{}_{kh} = O$.

(II)　**The K-connection.**　The curvature tensor with respect to this connection is already known as follows ([6], [9], [13]):

$$(3.8)_1\quad R^{a}_{b\,cd} = R_{j}{}^{i}{}_{kh}\,\zeta^{a}_{i}\zeta^{j}_{b}\zeta^{k}_{c}\zeta^{h}_{d}\,,\quad P^{a}_{b\,c(\sigma)} = P_{j}{}^{i}{}_{kh}\,\zeta^{a}_{i}\zeta^{j}_{b}\zeta^{k}_{c}\zeta^{h}_{\sigma}\,,\quad S^{a}_{b(\gamma)(\sigma)} = S_{j}{}^{i}{}_{kh}\,\zeta^{a}_{i}\zeta^{j}_{b}\zeta^{k}_{\gamma}\zeta^{h}_{\sigma}\,,$$

$$(3.8)_2\quad R^{a}_{(\beta)cd} = P^{a}_{(\beta)c(\sigma)} = S^{a}_{(\beta)(\gamma)(\sigma)} = O,$$

$$(3.8)_3\quad R^{(\alpha)}_{b}{}_{cd} = P^{(\alpha)}_{b}{}_{c(\sigma)} = S^{(\alpha)}_{b(\gamma)(\sigma)} = O,$$

$$(3.8)_4\quad R^{(\alpha)}_{(\beta)cd} = {}''R_{j}{}^{i}{}_{kh}\,\zeta^{\alpha}_{i}\zeta^{j}_{\beta}\zeta^{k}_{c}\zeta^{h}_{d}\,,\quad P^{(\alpha)}_{(\beta)c(\sigma)} = {}'P_{j}{}^{i}{}_{kh}\,\zeta^{\alpha}_{i}\zeta^{j}_{\beta}\zeta^{k}_{c}\zeta^{h}_{\sigma}\,,\quad S^{(\alpha)}_{(\beta)(\gamma)(\sigma)} = \widetilde{S}_{j}{}^{i}{}_{kh}\,\zeta^{\alpha}_{i}\zeta^{j}_{\beta}\zeta^{k}_{\gamma}\zeta^{h}_{\sigma}\,.$$

Suppose $K^{A}_{B\,CD} = O$.　Then from $(3.8)_1$ and $(3.8)_4$, we have $h_{jk}h^{i}_{h}$

$-h_{jh}h^i_k=0$, contraction of which by g^{ih} yields $(n-2)h_{jk}=0$. So we have $n=2$ or $h_{jk}=0$. A Finsler space M is called *quasi–locally Minkowskian* [14] if $R_j{}^i{}_{kh}=P_j{}^i{}_{kh}=0$. If $n=2$, we have always $S_j{}^i{}_{kh}=\widetilde{S}_j{}^i{}_{kh}=0$. Therefore, $K^A_{BCD}=0$ if and only if $R_j{}^i{}_{kh}=P_j{}^i{}_{kh}=0$.

Summarizing the results obtained, we have

Theorem 1. *The curvature tensor K^A_{BCD} with respect to the K_o–connection is given by* (3 . 7)$_1$~(3 . 7)$_4$. *In this case, $K^A_{BCD}=0$ if and only if any indicatrix of M is locally flat and ${}''R_j{}^i{}_{kh}={}'P_j{}^i{}_{kh}=0$. For the K–connection, the following hold good:*

(i) *When $n=2$, the curvature tensor K^A_{BCD} vanishes if and only if M is quasi–locally Minkowskian.*

(ii) *When $n\geqq 3$, the curvature tensor K^A_{BCD} never vanishes.*

(III) **The D–connection.** Because of (3 . 1)~(3 . 3) we can express as

$$(3.9)\quad \begin{aligned}\overline{\Omega}^a_b&=(\omega^c_b\wedge\omega^a_c-d\omega^a_b)+\omega^{(\gamma)}_b\wedge\omega^a_{(\gamma)}=\Omega^a_b+\sum_\varepsilon\omega^{(\varepsilon)}_a\wedge\omega^{(\varepsilon)}_b\\&=\frac{1}{2}\overline{R}_b{}^a{}_{cd}\,\omega^c\wedge\omega^d+\overline{P}_{bc(\sigma)}^a\,\omega^c\wedge\omega^{(\sigma)}+\frac{1}{2}\overline{S}_{b(\gamma)(\sigma)}^a\,\omega^{(\gamma)}\wedge\omega^{(\sigma)}.\end{aligned}$$

Therefore if we put

$$(3.10)_1\quad \begin{aligned}&\overline{R}_b{}^a{}_{cd}=\overset{1}{R}_{jikh}\,\zeta^j_b\zeta^i_a\zeta^k_c\zeta^h_d,\quad \overline{R}^a_{bc(\sigma)}=\overset{1}{P}_{jikh}\,\zeta^j_b\zeta^j_b\zeta^i_a\zeta^k_c\zeta^h_\delta,\\&\overline{S}^a_{b(\gamma)(\sigma)}=\overset{1}{S}_{jikh}\,\zeta^j_b\zeta^i_a\zeta^k_\gamma\zeta^h_\delta,\end{aligned}$$

then from (2 . 4), (3 . 3) and (3 . 9) we have

$$(3.10)_{1'}\quad \begin{aligned}&\overset{1}{R}_{jikh}=R_{jikh}+B_{rjh}B^r{}_{ik}-B_{rjk}B^r{}_{jh},\\&\overset{1}{P}_{jikh}=P_{jikh}+P_{rjh}B^r{}_{ik}-B_{rjk}P^r{}_{ih},\\&\overset{1}{S}_{jikh}=S_{jikh}+P_{rjh}P^r{}_{ik}-P_{rjk}P^r{}_{ih}.\end{aligned}$$

For $\overline{\Omega}^a_{(\beta)}$, we have

$$(3.11)\quad \begin{aligned}\overline{\Omega}^a_{(\beta)}&=d\omega^{(\beta)}_a+\sum_c\omega^a_c\wedge\omega^{(\beta)}_c+\sum_\gamma\omega^{(\gamma)}_a\wedge\omega^{(\gamma)}_{(\beta)}\\&=\frac{1}{2}\overline{R}_{(\beta)}{}^a{}_{cd}\,\omega^c\wedge\omega^d+\overline{P}_{(\beta)}{}^a{}_{c(\sigma)}\,\omega^c\wedge\omega^{(\sigma)}+\frac{1}{2}\overline{S}_{(\beta)}{}^a{}_{(\gamma)(\sigma)}\,\omega^{(\gamma)}\wedge\omega^{(\sigma)},\end{aligned}$$

$$(3.10)_2\quad \begin{aligned}&\overline{R}_{(\beta)}{}^a{}_{cd}=\overset{2}{R}_{jikh}\,\zeta^j_\beta\zeta^i_a\zeta^k_c\zeta^h_d,\quad \overline{P}_{(\beta)}{}^a{}_{c(\sigma)}=\overset{2}{P}_{jikh}\,\zeta^j_\beta\zeta^i_a\zeta^k_c\zeta^h_\delta,\\&\overline{S}_{(\beta)}{}^a{}_{(\gamma)(\sigma)}=\overset{2}{S}_{jikh}\,\zeta^j_\beta\zeta^i_a\zeta^k_\gamma\zeta^h_\delta.\end{aligned}$$

After a long caluculation using (1 . 14), (1 . 15), (1 . 17), (2 . 4), (3 . 11) and the Bianchi's identies, we have

$$(3.10)_{2'}\quad \begin{aligned}\overset{2}{R}_{jikh} &= B_{jik|h} - B_{jih|k} - P_{jir}R^{r}{}_{hk},\\ \overset{2}{P}_{jikh} &= B_{jik}|_{h} - P_{jih|k} + B_{hik}l_{j} + B_{jir}A^{r}{}_{kh} + P_{jir}P^{r}{}_{kh},\\ \overset{2}{S}_{jikh} &= P_{hjik} - P_{kjih} + P_{jih}l_{k} - P_{jik}l_{h} - P_{jkr}A^{r}{}_{ih} + P_{jhr}A^{r}{}_{ik}.\end{aligned}$$

For $\bar{\Omega}^{(\alpha)}_{b}$, we have

$$\bar{\Omega}^{(\alpha)}_{b} = -\Omega^{b}_{(\alpha)} = \frac{1}{2}\bar{R}^{(\alpha)}_{b\ cd}\,\omega^{c}\wedge\omega^{d} + \bar{P}^{(\alpha)}_{b\ c(\sigma)}\,\omega^{c}\wedge\omega^{(\sigma)} + \frac{1}{2}\bar{S}^{(\alpha)}_{b\ (\gamma)(\sigma)}\,\omega^{(\gamma)}\wedge\omega^{(\sigma)},$$

$$(3.10)_{3}\quad \bar{R}^{(\alpha)}_{b\ cd} = \overset{3}{R}_{j\ kh}^{\ i}\,\zeta^{\alpha}_{i}\zeta^{j}_{b}\zeta^{k}_{c}\zeta^{h}_{d},\quad \bar{P}^{(\alpha)}_{b\ c(\sigma)} = \overset{3}{P}_{j\ kh}^{\ i}\,\zeta^{\alpha}_{i}\zeta^{j}_{b}\zeta^{k}_{c}\zeta^{h}_{\sigma},$$

$$\bar{S}^{(\alpha)}_{b\ (\gamma)(\sigma)} = \overset{3}{S}_{j\ kh}^{\ i}\,\zeta^{\alpha}_{i}\zeta^{j}_{b}\zeta^{k}_{\gamma}\zeta^{h}_{\sigma},$$

$$(3.10)_{3'}\quad \begin{aligned}\overset{3}{R}_{j\ kh}^{\ i} &= B^{i}{}_{jk|h} - B^{i}{}_{jh|k} - P^{i}{}_{jr}R^{r}{}_{kh},\\ \overset{3}{P}_{j\ kh}^{\ i} &= P^{i}{}_{jh|k} - B^{i}{}_{jk}|_{h} - B_{hjk}l^{i} - B^{i}{}_{jr}A^{r}{}_{kh} - P^{i}{}_{jr}P^{r}{}_{kh},\\ \overset{3}{S}_{j\ kh}^{\ i} &= P_{k\ jh}^{\ i} - P_{h\ jk}^{\ i} - P^{i}{}_{jh}l_{k} + P^{i}{}_{jk}l_{h} - P^{i}{}_{hr}A^{r}{}_{jk} + P^{i}{}_{kr}A^{r}{}_{jh}.\end{aligned}$$

For $\bar{\Omega}^{(\alpha)}_{(\beta)}$, we have

$$(3.12)\quad \begin{aligned}\bar{\Omega}^{(\alpha)}_{(\beta)} &= \Omega^{(\alpha)}_{(\beta)} + \sum_{c}\omega^{(\alpha)}_{c}\wedge\omega^{(\beta)}_{c}\\ &= \frac{1}{2}\bar{R}_{(\beta)\ cd}^{\ (\alpha)}\,\omega^{c}\wedge\omega^{d} + \bar{P}_{(\beta)\ c(\sigma)}^{\ (\alpha)}\,\omega^{c}\wedge\omega^{(\sigma)}\\ &\quad + \frac{1}{2}S_{(\beta)\ (\gamma)(\sigma)}^{\ (\alpha)}\,\omega^{(\gamma)}\wedge\omega^{(\sigma)},\end{aligned}$$

$$(3.10)_{4}\quad \begin{aligned}&\bar{R}_{(\beta)\ cd}^{\ (\alpha)} = \overset{4}{R}_{j\ kh}^{\ i}\,\zeta^{\alpha}_{i}\zeta^{j}_{\beta}\zeta^{k}_{c}\zeta^{h}_{d},\quad \bar{P}_{(\beta)\ c(\sigma)}^{\ (\alpha)} = \overset{4}{P}_{j\ kh}^{\ i}\,\zeta^{\alpha}_{i}\zeta^{j}_{\beta}\zeta^{k}_{c}\zeta^{h}_{\sigma},\\ &\bar{S}_{(\beta)\ (\gamma)(\sigma)}^{\ (\alpha)} = \overset{4}{S}_{j\ kh}^{\ i}\,\zeta^{\alpha}_{i}\zeta^{j}_{\beta}\zeta^{k}_{\gamma}\zeta^{h}_{\sigma}.\end{aligned}$$

From (2 . 4), (3 . 4), (3 . 5) and (3 . 12) we obtain

$$(3.10)_{4'}\quad \begin{aligned}\overset{4}{R}_{j\ kh}^{\ i} &= {}''R_{j\ kh}^{\ i} + (B^{i}{}_{sk}B_{jth} - B^{i}{}_{sh}B_{jtk})g^{st},\\ \overset{4}{P}_{j\ kh}^{\ i} &= {}'P_{j\ kh}^{\ i} + (B^{i}{}_{sk}P_{jth} - P^{i}{}_{sh}B_{jtk})g^{st},\\ \overset{4}{S}_{j\ kh}^{\ i} &= \widetilde{S}_{j\ kh}^{\ i} + P^{i}{}_{rk}P^{r}{}_{jh} - P^{i}{}_{rh}P^{r}{}_{jk}.\end{aligned}$$

Especially if $\overset{1}{S}_{jikh} = \overset{4}{S}_{jikh} = 0$, then we have $(n-2)h_{jk} = 0$. Therefore when $n \geqq 3$, *the tensor* K^{A}_{BCD} never vanishes.

(Ⅳ) **The D_o−connection.** In the same way as before we can express as

$$\overline{\Omega}{}_b^a = \frac{1}{2}\overline{R}{}_{b\,cd}^{\,a}\,\omega^c\wedge\omega^d + \overline{P}{}_{b\,c(\sigma)}^{\,a}\,\omega^c\wedge\omega^{(\sigma)} + \frac{1}{2}\overline{S}{}_{b(\gamma)(\sigma)}^{\,a}\,\omega^{(\gamma)}\wedge\omega^{(\sigma)},$$

and put

$$(3.11)_1 \qquad \begin{aligned} &\overline{R}{}_{b\,cd}^{\,a} = \overset{(1)}{R}{}_{j\,kh}^{\,i}\,\zeta_i^a\zeta_b^j\zeta_c^k\zeta_d^h, \quad \overline{P}{}_{b\,c(\sigma)}^{\,a} = \overset{(1)}{P}{}_{j\,kh}^{\,i}\,\zeta_i^a\zeta_b^j\zeta_c^k\zeta_\sigma^h, \\ &\overline{S}{}_{b(\gamma)(\sigma)}^{\,a} = \overset{(1)}{S}{}_{j\,kh}^{\,i}\,\zeta_i^a\zeta_b^j\zeta_\gamma^k\zeta_\sigma^h. \end{aligned}$$

In this case, since the relation between the D_o- and $D-$connectinos is the same as that between the K_o- and $K-$connections, we have

$$(3.11)_{1'} \qquad \begin{aligned} &\overset{(1)}{R}_{jikh} = {}''R_{jikh} + B_{rjh}B^r{}_{ik} - B_{rjk}B^r{}_{ih}, \\ &\overset{(1)}{P}_{jikh} = {}'P_{jikh} + P_{rjh}B^r{}_{ik} - B_{rjk}P^r{}_{ih}, \\ &\overset{(1)}{S}_{jikh} = \widetilde{S}_{jikh} + P_{rjh}P^r{}_{ik} - P_{rjk}P^r{}_{ih}. \end{aligned}$$

For $\overline{\Omega}{}_{(\beta)}^a$, we have

$$\begin{aligned} \overline{\Omega}{}_{(\beta)}^a &= \overline{\Omega}{}_{(\beta)}^a - \omega_n^a\wedge\omega_n^{(\beta)} = \overline{\Omega}{}_{(\beta)}^a - (h_h^i R_{jok})\,\zeta_i^a\zeta_\beta^j\zeta_c^k\zeta_\sigma^h\,\omega^c\wedge\omega^{(\sigma)} \\ &= \frac{1}{2}\overline{R}{}_{(\beta)\,cd}^{\,a}\,\omega^c\wedge\omega^d + \overline{P}{}_{(\beta)\,c(\sigma)}^{\,a}\,\omega^c\wedge\omega^{(\sigma)} + \overline{S}{}_{(\beta)(\gamma)(\sigma)}^{\,a}\,\omega^{(\gamma)}\wedge\omega^{(\sigma)}, \end{aligned}$$

$$(3.11)_2 \qquad \begin{aligned} &\overline{R}{}_{(\beta)\,cd}^{\,a} = \overset{(2)}{R}_{jikh}\,\zeta_\beta^j\zeta_a^i\zeta_c^k\zeta_d^h, \quad \overline{P}{}_{(\beta)\,c(\sigma)}^{\,a} = \overset{(2)}{P}_{jikh}\,\zeta_\beta^j\zeta_a^i\zeta_c^k\zeta_\sigma^h, \\ &\overline{S}{}_{(\beta)(\gamma)(\sigma)}^{\,a} = \overset{(2)}{S}_{jikh}\,\zeta_\beta^j\zeta_a^i\zeta_\gamma^k\zeta_\sigma^h, \end{aligned}$$

$$(3.11)_{2'} \qquad \overset{(2)}{R}_{jikh} = \overset{2}{R}_{jikh}, \quad \overset{(2)}{P}_{jikh} = \overset{2}{P}_{jikh} + h_{ih}R_{jok}, \quad \overset{(2)}{S}_{jikh} = \overset{2}{S}_{jikh}.$$

For $\overline{\Omega}{}_b^{(\alpha)}$, we have

$$(3.11)_3 \qquad \begin{aligned} &\overline{R}{}_b^{(\alpha)}{}_{cd} = \overset{(3)}{R}{}_{j\,kh}^{\,i}\,\zeta_i^\alpha\zeta_b^j\zeta_c^k\zeta_d^h, \quad \overline{P}{}_b^{(\alpha)}{}_{c(\sigma)} = \overset{(3)}{P}{}_{j\,kh}^{\,i}\,\zeta_i^\alpha\zeta_b^j\zeta_c^k\zeta_\sigma^h, \\ &\overline{S}{}_{b(\gamma)(\sigma)}^{(\alpha)} = \overset{(3)}{S}{}_{j\,kh}^{\,i}\,\zeta_i^\alpha\zeta_b^j\zeta_\gamma^k\zeta_\sigma^h, \end{aligned}$$

$$(3.11)_{3'} \qquad \overset{(3)}{R}{}_{j\,kh}^{\,i} = \overset{3}{R}{}_{j\,kh}^{\,i}, \quad \overset{(3)}{P}{}_{j\,kh}^{\,i} = \overset{3}{P}{}_{j\,kh}^{\,i} + h_{jh}R^i{}_{ok}, \quad \overset{(3)}{S}{}_{j\,kh}^{\,i} = \overset{3}{S}{}_{j\,kh}^{\,i}.$$

For $\overline{\Omega}{}_{(\beta)}^{(\alpha)}$, we have

$$(3.11)_4 \qquad \overline{S}{}_{(\beta)(\gamma)(\sigma)}^{(\alpha)} = \overset{(4)}{S}{}_{j\,kh}^{\,i}\,\zeta_i^\alpha\zeta_\beta^j\zeta_\gamma^k\zeta_\sigma^h,$$

$$(3.11)_{4'} \qquad \overset{(4)}{R}{}_{j\,kh}^{\,i} = \overset{4}{R}{}_{j\,kh}^{\,i}, \quad \overset{(4)}{P}{}_{j\,kh}^{\,i} = \overset{4}{P}{}_{j\,kh}^{\,i}, \quad \overset{(4)}{S}{}_{j\,kh}^{\,i} = \overset{4}{S}{}_{j\,kh}^{\,i}.$$

Thus we have

Theorem 2. *The curvature tensor K^A_{BCD} with respect to the $D-$connection is given by $(3.10)_1$, $(3.10)_{1'}$ $\sim (3.10)_4$, $(3.10)_{4'}$. Especially* *when* $n \geqq 3$, *the tensor* K^{A}_{BCD} *never vanishes. The curvature tensor* K^{A}_{BCD} *with respect to the* D_0*-connection is given by* $(3.11)_1$, $(3.11)_{1'} \sim (3.11)_4$, $(3.11)_{4'}$.

§4. **Special cases.** In the previous section we have introduced many new tensors on M. However, for the present, we can not explicate the geometrical meanings of these tensors. In this section, therefore, we shall try it for special Finsler spaces.

(A) Let M be a space with absolute parallelism of line-elements, that is, $R^{i}{}_{jk}=0$. Then for the D-connection, we have

$$
\begin{aligned}
&\overset{1}{R}_{jikh}=\overset{4}{R}_{jikh}=R_{jikh}+S_{jikh}\,,\\
&\overset{1}{P}_{jikh}=P_{jikh}+P_{rjh}A^{r}{}_{ik}-A_{rjk}P^{r}{}_{ih}\,,\\
&\overset{1}{S}_{jikh}=S_{jikh}+P_{rjh}P^{r}{}_{jk}-P_{rjk}P^{r}{}_{ih}\,,\\
&\overset{2}{R}_{jikh}=\overset{2}{R}_{ijkh}=-\overset{3}{R}_{ijkh}=-\overset{3}{R}_{jikh}=+A_{jik|h}-A_{jih|k}\,,\\
(4.1)\qquad &\overset{2}{P}_{jikh}=-\overset{3}{P}_{ijkh}=+A_{jik}|_{h}-P_{jih|k}+A_{hik}l_{j}+A_{jir}A^{r}{}_{kh}+P_{jir}P^{r}{}_{kh}\,,\\
&\overset{2}{S}_{jikh}=\overset{2}{S}_{ijkh}=-\overset{3}{S}_{ijkh}=-\overset{3}{S}_{jikh}=+P_{hjik}-P_{kjih}\\
&\qquad -P_{jkr}A^{r}{}_{ih}+P_{jhr}A^{r}{}_{ik}-P_{jik}l_{h}+P_{jih}l_{k}\,,\\
&\overset{4}{P}_{jikh}={}'P_{jikh}+A_{rjh}P^{r}{}_{ik}-A_{rjk}P^{r}{}_{ih}\,,\\
&\overset{4}{S}_{jikh}=\widetilde{S}_{jikh}+P_{rjh}P^{r}_{ik}-P_{rjk}P^{r}_{ih}
\end{aligned}
$$

Then we have

Theorem 3. *Let* M *be a space with absolute parallelism of line-elements. Then the curvature tensor* K^{A}_{BCD} *with respect to the D—connection is determined by* $(4\cdot 1)$. *In this case, the following propositions hold good:*

(1) *M is quasi-locally Minkowskian if and only if* $\overset{1}{P}_{jikh}=0$ *or* $\overset{2}{R}_{jikh}=0$.

(2) *M is locally Euclidean if and only if* $\overset{2}{P}_{jikh}=0$ *and* ($\overset{1}{P}_{jikh}=0$ *or* $\overset{2}{R}_{jikh}=0$).

(3) *When* $n=2$, *the curvature tensor* K^{A}_{BCD} *vanishes if and only if M is locally Euclidean.*

Proof. Firstly, we shall prove the proposition (1). Contracting $\overset{1}{P}_{jikh}=0$ by l^{j}, we have $P_{ikh}=0$. We know that $P_{ijk}=R_{ijk}=0$ is equivalent to $P_{jikh}=R_{jikh}=O$. Next, it is well known that $P_{ijk}=O$ is equivalent to $\overset{2}{R}_{jikh}=A_{jih|k}-A_{jik|h}=O$.

Secondly, we prove the proposition (2). Suppose $P_{ijk}=O$. Then from $\overset{2}{P}_{jikh}=O$, we obtain

$$(4.2)\qquad A_{jik}|_h=-l_j A_{ikh}-A_{jir}A^r{}_{kh}.$$

Since the right hand side of (4.2) is symmetric in the indices k and h, we have $A_{jik}|_h-A_{jih}|_k=O$, contraction of which by l^k yields $A_{jih}=O$, that is, M is Riemannian. Accordingly $R_h{}^i{}_{jk}$ are functions of x alone. Then, differentiating $R_h{}^i{}_{jk}y^h=O$ by y^h, we have $R_h{}^i{}_{jk}=O$.

Lastly, we prove the proposition (3). If M is of dimension 2, then we have always $S_{jikh}=\widetilde{S}_{jikh}=O$. If $K^A_{BCD}=O$, it follows from the proposition (2) that M is locally Euclidean. The converse is clear. Q. E. D.

For the D_o-connection, we have

$$(4.3)\qquad \begin{aligned} &\overset{(1)}{R}_{jikh}=\overset{(4)}{R}_{jikh}=\overset{1}{R}_{jikh},\quad \overset{(1)}{P}_{jikh}=\overset{(4)}{P}_{jikh}=\overset{4}{P}_{jikh},\\ &\overset{(1)}{S}_{jikh}=\overset{(4)}{S}_{jikh}=\overset{4}{S}_{jikh},\quad \overset{(2)}{R}_{jikh}=-\overset{(3)}{R}_{jikh}=\overset{2}{R}_{jikh},\\ &\overset{(2)}{P}_{jikh}=-\overset{(3)}{P}_{jikh}=\overset{2}{P}_{jikh},\quad \overset{(2)}{S}_{jikh}=-\overset{(3)}{S}_{ijkh}=\overset{2}{S}_{jikh}. \end{aligned}$$

Hence we have

Corollary 3.1. *Let M be a space with absolute parallelism of line-elements. Then the curvature tensor K^A_{BCD} with respect to the D_o-connection is dertermined by only the components $\overset{1}{R}_{jikh}$, $\overset{2}{R}_{jikh}$, $\overset{2}{P}_{jikh}$, $\overset{2}{S}_{jikh}$, $\overset{4}{P}_{jikh}$ and $\overset{4}{S}_{jikh}$ of the curvature tensor with respect to the D-connection. And the following propositions hold good still:*

(1) *M is quasi-locally Minkowskian if and only if $\overset{(2)}{R}_{jikh}=O$.*

(2) *M is locally Euclidean if and only if $\overset{(2)}{R}_{jikh}=\overset{(2)}{P}_{jikh}=O$.*

(3) *When $n=2$, the curvature tensor K^A_{BCD} vanishes if and only if M is locally Euclidean.*

(B) Let M be a Landsberg space, that is, $P^i{}_{jk}=O$. Then for the D-connection, we have

$$\overset{1}{P}_{jikh}=\overset{2}{S}_{jikh}=\overset{3}{S}_{jikh}=\overset{4}{P}_{jikh}=O,\quad \overset{1}{S}_{jikh}=S_{jikh},$$

$$\overset{1}{R}_{jikh} = P_{jikh} + B_{rjh} B^{r}{}_{ik} - B_{rjk} B^{r}{}_{ih}, \quad \overset{4}{S}_{jikh} = \widetilde{S}_{jikh},$$

$$(4.4) \qquad \overset{2}{R}_{jikh} = -\overset{3}{R}_{ijkh} = +R_{jik|h} - R_{jih|k}$$

$$\overset{2}{P}_{jikh} = -\overset{3}{P}_{ijkh} = B_{jik}|_{h} + B_{hik} l_{j} + B_{jir} A^{r}{}_{kh},$$

$$\overset{4}{R}_{jikh} = {}^{\prime\prime}R_{jikh} + (B_{isk} B_{jth} - B_{ish} B_{jtk}) g^{st}.$$

Then we can state

Theorem 4. *Let M be a Landsberg space. Then the curvature tensor* K^{A}_{BCD} *with respect to the D–connection is determined by* (4.4). *In this case, the following propositions hold good:*

(1) *The tensor* R_{ijk} *is h–covariant constant on M if and only if* $\overset{2}{R}_{jikh} = O$.

(2) *When* $n \geqq 3$, *M is locally Euclidean if and only if* $\overset{2}{P}_{jikh} = O$. *When* $n = 2$, *M is a Riemannian space if and only if* $\overset{2}{P}_{jikh} = O$.

(3) *If M is of scalar curvature and* $\overset{1}{R}_{jikh} = O$, *then M is a quasi-locally Minkowski space with* $S_{jikh} = O$ *or M is a Riemannian space of dimension 2 or M is locally Euclidean.*

(4) *If M is of constant curvature and* $\overset{1}{R}_{jikh} = O$, *then M is a quasi–locally Minkowski space with* $S_{jikh} = O$ *or M is a two–dimensional Riemannian space of constant curvature or M is a Riemannian space of constant curvature 1.*

Proof. Firstly we prove the proposition (1). In a Landsberg space, a Bianchi's identity $R_{jik|h} + R_{jhi|k} + R_{jkh|i} = O$ holds. From this identity and $\overset{2}{R}_{jikh} = R_{jik|h} - R_{jih|k} = O$, we have $R_{jkh|i} = O$.

Secondly, let us prove the proposition (2). We have another identity

$$(4.5) \qquad R_{jik}|_{h} + R_{jik} l_{h} - R_{hjik} + R_{jir} A^{r}{}_{kh} + R_{jrk} A^{r}{}_{ih} = O.$$

Suppose $\overset{2}{P}_{jikh} = O$, that is,

$$(4.6) \qquad B_{jik}|_{h} + B_{hik} l_{j} + B_{jir} A^{r}{}_{kh} = O.$$

Then subtracting (4.5) from (4.6), we have

$$(4.7) \qquad A_{jik}|_{h} + A_{jir} A^{r}{}_{kh} + R_{hjik} - R_{jik} l_{h} + R_{hik} l_{j} + A_{hik} l_{j} + R_{jkr} A^{r}{}_{ih} = O.$$

Since $A_{jik}|_h + A_{jir}A^r{}_{kh}$ is symmetric in the indices j and i, from (4 . 7) we have

$$R_{hjik} - R_{jik}l_h + R_{hik}l_j + A_{hik}l_j + R_{jkr}A^r{}_{jh}$$
$$= R_{hijk} - R_{ijk}l_h + R_{hjk}l_i + A_{hjk}l_i + R_{ikr}A^r{}_{jh},$$

contraction of which by l^j yields

(4 . 8) $\quad A_{hik} = R_{hiok} - R_{iok}l_h + R_{hok}l_i\,.$

The left hand side of (4 . 8) is symmetric in the indices h and i, while the right hand side is skew-symmetric in the same indices. Therefore we have $A_{hik} = O$ and hence from (4 . 7)

(4 . 9) $\quad R_{hjik} = R_{jik}l_h - R_{hik}l_j\,,$

which implies

(4 . 10) $\quad F^2 R_{hjik} = R_{sjik}y^s g_{ht}y^t - R_{shik}y^s g_{jt}y^t\,.$

Differentiating (4 . 10) by y^p and y^q two times, contracting the resulting expression by g^{qs} and summing the result with respect to s and p, we obtain $(n-2)R_{hjik} = O$ i.e. $R_{hjik} = O$ or $n = 2$. Conversely if $A_{ijk} = O$ and $R_{hjik} = O$, then $\overset{2}{P}_{jikh} = O$. When $n = 2$, the relation (4 . 9) always holds. Therefore if $A_{ijk} = O$, then from (4 . 5) and (4 . 9) we have $\overset{2}{P}_{jikh} = R_{jik}|_h + R_{hik}l_j = O$.

Thirdly, we prove the proposition (3). From $\overset{1}{R}_{jikh} = O$, we have

(4 . 11) $\quad R_{jikh} + S_{jikh} + A_{rjh}A^r{}_{ik} - A_{rjk}R^r{}_{ih} + R_{rjh}A^r{}_{ik} - R_{rjk}A^r{}_{ih}$

$$+ R_{rjh}R^r{}_{ik} - R_{rjk}R^r{}_{ih} = O\,,$$

contraction of which by l^j yields

(4 . 12) $\quad R_{ikh} + R_{roh}A^r{}_{ik} - R_{rok}A^r{}_{ih} + R_{roh}R^r{}_{ik} - R_{rok}R^r{}_{ih} = O.$

Further contracting (4 . 12) by l^k, we have

(4 . 13) $\quad R_{ioh} = R_{roh}R^r{}_{oi}\,.$

Let M be of scalar curvature R, that is, (4 . 14) $\qquad R_{ioh}=R(x,y)\,h_{ih}\,,$

where $R(x,y)$ is a homogeneous function of degree O in y^i . If we subsitute (4.14) in (4.13), we get $R(R-1)\,h_{ih}=O$, i. e. $R=O$ or $R=1$. If $R=O$, then from (4 . 12) and (4 . 13) we have $R_{ikh}=O$ and hence $R_{jikh}=O$. Therefore (4 . 11) leads us to $S_{jikh}=O$.

Next, suppose $R=1$. Then we can express R_{ikh} as

(4 . 15) $\qquad R_{ikh}=l_k g_{ih}-l_h g_{ik}\,.$

Substituting (4 . 15) in (4 . 11), we have

(4 . 16) $\qquad R_{jikh}=l_j R_{ikh}-l_i R_{jkh}-S_{jikh}\,.$

Again if we substitute (4 . 15) and (4 . 16) in (4 . 5), then we get

$$h_{ih}\,g_{ik}-h_{kh}\,g_{ji}+l_i R_{hik}+S_{hjik}+l_i A_{jkh}-l_k A_{jih}=O,$$

constraction of which by l^i gives $A_{jkh}=O$. The sequent proof is the same as in the proposition (2).

Lastly we prove the proposition (4). From $\overset{4}{R}_{jikh}=O$, we have

(4 . 17) $$R_{jikh}-l_j R_{ikh}+l_i R_{jkh}+S_{jikh}+(A_{isk}R_{jth}-A_{ish}R_{jtk}$$
$$+R_{isk}A_{jth}-R_{ish}A_{jtk}+R_{isk}R_{jth}-R_{ish}R_{jtk})\,g^{st}=O.$$

Let M be of constant curvature, i. e.

(4 . 18) $\qquad R_{ijk}=R(l_j g_{ik}-l_k g_{ij})$, where R is constant.

Substituting (4 . 18) in (4 . 5) and (4 . 17), we have

(4 . 19) $\qquad R_{jikh}=l_j R_{ikh}+R(h_{jk}g_{ih}-h_{jh}g_{ik}+l_k A_{ihj}-l_h A_{ikj}),$

(4 . 20) $$R_{jikh}+S_{jikh}-l_j R_{ikh}+l_i R_{jkh}+R(l_h R_{jik}-l_k R_{jih})$$
$$+R^2(h_{jh}g_{ik}-h_{jk}g_{ih})=O.$$

Again, substitution of (4 . 19) in (4 . 20) yields

(4 . 21) $$S_{jikh}+l_i R_{jikh}+R(h_{jk}g_{ih}-h_{jh}g_{ik}+l_h R_{jik}-l_k R_{jih}$$
$$+l_k A_{ijh}-l_h A_{ijk})+R^2(h_{jh}g_{ik}-h_{jk}g_{ih})=O,$$

contraction of which by l^k gives $RA_{jih}=O$, i. e. $R=O$ or $A_{jih}=O$.

If $R=O$, then we have $R_{jikh}=S_{jikh}=O$. If $A_{jih}=O$, then (4 . 21) leads us to $R=1$ or

$$h_{jh}g_{ik}-h_{jk}g_{ih}+l_i(l_h g_{jk}-l_k g_{jh})=O. \tag{4 . 22}$$

Contracting (4 . 22) by g^{ih}, we have $(n-2)h_{jk}=O$, i. e. $n=2$. Q. E. D.

For the D_o-connection, we have

$$\begin{aligned}
&\overset{(1)}{P}_{jikh}=\overset{(2)}{S}_{jikh}=\overset{(3)}{S}_{jikh}=\overset{(4)}{P}_{jikh}=O,\ \overset{(1)}{S}_{jikh}=\overset{(4)}{S}_{jikh}=\widetilde{S}_{jikh},\\
&\overset{(1)}{R}_{jikh}={}''R_{jikh}+B_{rjh}B^r{}_{ik}-B_{rjk}B^r{}_{ih},\ \overset{(2)}{R}_{jikh}=-\overset{(3)}{R}_{ijkh}=\overset{2}{R}_{jikh},\\
&\overset{(2)}{P}_{jikh}=-\overset{(3)}{P}_{ijkh}=B_{jik}|_h+B_{hik}l_j+B_{jir}A^r{}_{kh}-h_{ih}R_{jok},\\
&\overset{(4)}{R}_{jikh}(={}''R_{jikh}+(B_{isk}B_{jth}-B_{ish}B_{jtk})g^{st})=\overset{4}{R}_{jikh}.
\end{aligned} \tag{4 . 22}$$

Then we have

Corollary 4. 4. *Let M be a Landsberg space. Then the curvature tensor* K^A_{BCD} *with respect to the* D_o*-connection is determined by (4 . 22). In this case, the following propositions hold good:*

(1) *The tensor* R_{ijk} *is h-covariant constant on M if and only if* $\overset{(2)}{R}_{jikh}=O$

(2) *M is locally Euclidean if and only if* $\overset{(2)}{R}_{jikh}=O$.

(3) *If M is of scalar curvature and* $\overset{(1)}{R}_{jikh}=O$, *the M is a quasi-locally Minkowski space with* $S_{jikh}=O$.

(4) *If M is of constant curvature and* $\overset{(4)}{R}_{jikh}=O$, *then M is a quasi-locally Minkowski space with* $S_{jikh}=O$ *or M is a two-dimensional Riemannian space of constant curvature or M is a Riemannian space of constant curvature 1.*

Proof. The propositions (1) and (4) are the same as in Theorem 4. First, we prove the proposition (2). From $\overset{(2)}{P}_{jikh}=O$, we have

$$B_{jik}|_h+B_{hik}l_j+B_{jir}A^r{}_{kh}-h_{ih}R_{jok}=O. \tag{4 . 23}$$

Then subtracting (4 . 5) from (4 . 23), we obtain

(4 . 24) $$A_{jik}|_h + A_{jir} A^r{}_{kh} + R_{hjik} - R_{jik} l_h + R_{hik} l_j + A_{hik} l_j + R_{jkr} A^r{}_{ih} - h_{ih} R_{jok} = O.$$

In the same way as in Theorem 4, we have

(4 . 25) $$A_{hik} = R_{hiok} - R_{iok} l_h + R_{hok} l_i = O.$$

Therefore from (4 . 24) we get

(4 . 26) $$R_{hjik} = R_{jik} l_h - R_{hik} l_j + h_{ih} R_{jok},$$

contraction of which by g^{ih} yields

(4 . 27) $$R_{jk} = R_{ok} l_j + nR_{jok},$$

which implies

(4 . 28) $$F^2 R_{jk} = R_{ik} y^i g_{jh} y_h + nR_{ijhk} y^i y^h.$$

Differentiating (4 . 28) by y^i and y^h two times, we have

(4 . 29) $$2g_{ih} R_{jk} = R_{ik} g_{jh} + R_{hk} g_{ij} + n(R_{ijhk} + R_{hjik}),$$

contraction of which by g^{ih} gives $R_{jk} = O$. Then, from (4 . 27) and (4 . 29) we have

(4 . 30) $$R_{jok} = O, \quad R_{ijhk} + R_{hjik} = O.$$

In the same way as before, we have $R_{jikh} = O$ or $n = 2$. When $n = 2$, the condition (4 . 30) implies $R_{jikh} = O$, too.

Next, we prove the proposition (3). From $\overset{(1)}{R}_{jikh} = O$, we obtain

(4 . 31) $$R_{jikh} - l_j R_{ikh} + l_i R_{jkh} + S_{jikh} + A_{rjh} R^r{}_{ik} - A_{rjk} R^r{}_{ih} + R_{rjh} A^r{}_{ik} - R_{rjk} A^r{}_{ih} + R_{rjh} R^r{}_{ik} - R_{rjk} R^r{}_{ih} = O,$$

contraction of which by l^k yields

(4 . 32) $$R_{jioh} - l_j R_{ioh} + l_i R_{joh} - A_{rjh} R^r{}_{oi} + R_{roj} A^r{}_{ih} - R_{rjh} R^r{}_{oi} + R_{roj} R^r{}_{ih} = O.$$

Further contracting (4 . 32) by l^j, we have

(4 . 33) $\quad R_{roh} R^r{}_{oi} = O.$

Let M be of scalar curvature. Then if we substitute (4 . 14) in (4 . 33), then we have $R=O$ and hence $R_{ioh}=O$. Therefore by virtue of (4 . 32) we have $R_{jioh}=O$, from which it follows that

(4 . 34) $\quad R_{jikh} = -R_{jiks}|_h l^s .$

On the other hand, the following identity holds good:

(4 . 35) $\quad R_{jiks}|_h + S_{jisr} R^r{}_{hk} - R_{jisr} A^r{}_{kh} + R_{jikr} A^r{}_{sh} = O.$

In consequence of (4 . 34), (4 . 35) and $R_{jioh}=O$, we have $R_{jikh}=O$. Accordingly from (4 . 31) we obtain $S_{jikh}=O$. Q. E. D.

(C) Let M be a quasi–locally Minkowski space, that is, $P^i{}_{jk} = R^i{}_{jk} = O$. Then for the D–connection, we obtain

(4 . 36)
$$\overset{1}{P}_{jikh} = \overset{2}{S}_{jikh} = \overset{3}{S}_{jikh} = \overset{2}{R}_{jikh} = \overset{3}{R}_{jikh} = \overset{4}{P}_{jikh} = O,$$
$$\overset{1}{R}_{jikh} = \overset{4}{R}_{jikh} = \overset{1}{S}_{jikh} = S_{jikh}, \quad \overset{4}{S}_{jikh} = \widetilde{S}_{jikh},$$
$$\overset{2}{P}_{jikh} = -\overset{3}{P}_{ijkh} = A_{jik}|_h + A_{hik} l_j + A_{jir} A^r{}_{kh}.$$

For the D_o–connection, we have

(4 . 37)
$$\overset{(1)}{P}_{jikh} = \overset{(2)}{S}_{jikh} = \overset{(3)}{S}_{jikh} = \overset{(2)}{R}_{jikh} = \overset{(3)}{R}_{jikh} = \overset{(4)}{P}_{jikh} = O,$$
$$\overset{(1)}{R}_{jikh} = \overset{(4)}{R}_{jikh} = S_{jikh}, \quad \overset{(1)}{S}_{jikh} = \overset{(4)}{S}_{jikh} = \widetilde{S}_{jikh},$$
$$\overset{(2)}{P}_{jikh} = -\overset{(3)}{P}_{ijkh} = \overset{2}{P}_{jikh}.$$

Then we have

Proposition 4. *Let M be a quasi–locally Minkowski space. Then the curvature tensors with respect to the D_o– and D–connections are determined by (4 . 36) and (4 . 37) respectively. M is locally Euclidean if and only if $\overset{2}{P}_{jikh} = O$.*

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