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A Study of Special Subspaces in a Finsler Space ※)

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Introduction. In Riemannian geometry, the following two theorems are well known:

Theorem A. If an n-dimensional Riemannian space M_n is of constant curvature R and an m-dimensional subspace M_m of M_n is totally geodesic then M_m is also of constant curvature R.

Theorem B. If M_n is of constant curvature R and M_m is totally umbilical then M_m is also of constant curvature.

The principal purpose of the present paper is to study how the above theorems are generalized to Finsler geometry.

The terminologies and notations refer to the papers [5]〜[7] [1] unless otherwise stated.

§1. Preliminaries. Let M_n be an n-dimensional Finsler space with a fundamental function $L(x^i, y^i)$ and be endowed with a Matsumoto connection (usually called a Finsler connection [2]) $M\Gamma = (\Gamma_j{}^i{}_k, \Gamma^i{}_k, \widetilde{C}_j{}^i{}_k)$ This is a quite general connection with no metrical property and is defined as follows ([5], [6], [7]): The

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1) Numbers in blackets refer to the references at the end of the paper.

v-connection $\widetilde{C}_j{}^i{}_k$ is any $(-1)p$-homogeneous[2] tensor and $\Gamma^i{}_k$, $\Gamma_j{}^i{}_k$ are given by

$$(1.1)\qquad \Gamma^i{}_k = G^i{}_k + T^i{}_k,\quad \Gamma_j{}^i{}_k = \Gamma^i{}_{k\|j} + Q_j{}^i{}_k = G_j{}^i{}_k + T_j{}^i{}_k + Q_j{}^i{}_k,$$

where the symbol $\|j$ indicates the partial differentiation by y^j, $G^i{}_k$ and $G_j{}^i{}_k$ $(=G^i{}_{k\|j})$ are the non-linear connection and h-connection of Berwald, $T^i{}_k$ and $Q_j{}^i{}_k$ are $(1)p$-and $(0)p$-homogeneous tensors respectively and $T_j{}^i{}_k = T^i{}_{k\|j}$.

Let M_m be an m-dimensional subspace of M_n defined by

$$(1.2)\qquad x^i = x^i(u^\alpha)\ (i=1,2,3,\ldots\ldots n\ ;\ \alpha=1,2,3,\ldots\ldots\ldots,m),$$

provided that variables u^α form a coordinate system of M_m and the matrix with components $B^i{}_\alpha(=\partial x^i/\partial u^\alpha)$ is of rank m.

If we denote the components of a vector y^i tangent to a curve in M_m by y^α[3] in terms of u^α−system, then we have

$$(1.3)\qquad y^i = B^i{}_\alpha y^\alpha,\quad y^i{}_{\|\alpha}(=\partial y^i/\partial y^\alpha) = B^i{}_\alpha.$$

By means of the metric tensor $g_{ij}\ (=\frac{1}{2}L^2{}_{\|i\|j})$, we choose $n-m$ normal vectors $N^i_a\ (a=m+1,\ldots\ldots,n)$ at each point (u^α) of M_m as follows:

$$(1.4)\qquad g_{ij}N^i_a N^j_b = \delta_{ab},\quad B^i{}_\alpha N^b_i = 0,\quad N^b_j := g_{ij}N^i_b.$$

The induced fundamental function $\bar{L}(u^\alpha, y^\alpha)$ and metric tensor $g_{\beta\gamma}(u^\alpha, y^\alpha)$ on M_m are given by

$$(1.5)\qquad \bar{L} = L(x^i(u^\alpha), B^i{}_\beta y^\beta),\quad g_{\beta\gamma} = g_{jk}B^{jk}_{\beta\gamma},\quad B^{jk}_{\beta\gamma} := B^j{}_\beta B^k{}_\gamma.$$

Let g^{jk} and $g^{\beta\gamma}$ be the reciprocal tensors of g_{jk} and $g_{\beta\gamma}$. We put

$$C_{ijk} = \frac{1}{2}g_{ij\|k},\quad C_j{}^i{}_k = C_{jsk}g^{si},\quad B_i{}^\beta = g_{ij}B^j{}_\gamma g^{\gamma\beta},\quad \lambda^a_{b\gamma} = -N^a_i N^i_{b\|\gamma}.$$

In this case, we have the following relations:

$$(1.6)\qquad \begin{aligned} &B_i{}^\beta{}_{\|\gamma} = C_b{}^\beta{}_\gamma N^b_i,\quad N^i_{b\|\gamma} = -2C_b{}^\beta{}_\gamma B^i{}_\beta - \lambda^c_{b\gamma}N^i_c,\\ &N^b_i{}_{\|\gamma} = \lambda^b{}_{c\gamma}N^c_i,\quad \lambda^b_{c\gamma} + \lambda^c_{b\gamma} = 2C_b{}^c{}_\gamma = 2C_c{}^b{}_\gamma,\end{aligned}$$

where $C_b{}^\beta{}_\gamma = C_j{}^i{}_k B_i{}^\beta B^j{}_\gamma N^k_b$ and $C_b{}^c{}_\gamma = C_j{}^i{}_k N^j_b N^c_i B^k{}_\gamma$.

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2) "$(r)p$-homogeneous" means "positively homogeneous of degree r in y^i"

3) If no confusion occurs then we use y^α in stead of the usual notation v^α.

The induced Matsumoto connection $IM\Gamma = (\Gamma_\beta{}^a{}_\gamma, \Gamma^a{}_\gamma, \widetilde{C}_\beta{}^a{}_\gamma)$ on M_m is defined as follows [5]:

$$(1.7)\quad \begin{aligned} \Gamma_\beta{}^a{}_\gamma &= B_i{}^a(B_\beta{}^i{}_\gamma + \Gamma_j{}^i{}_k B^{jk}_{\beta\gamma}) + \widetilde{C}_\beta{}^a{}_b H^b_\gamma\,, \\ \Gamma^a{}_\gamma &= B_i{}^a(B_0{}^i{}_\gamma + \Gamma^i{}_k B^k{}_\gamma),\quad \widetilde{C}_\beta{}^a{}_\gamma = \widetilde{C}_j{}^i{}_k B_i{}^a B^{jk}_{\beta\gamma}\,, \end{aligned}$$

where $B_\beta{}^i{}_\gamma = \partial B^i{}_\gamma/\partial u^\beta$, $B_0{}^i{}_\beta = B_\beta{}^i{}_\gamma y^\beta$, $\widetilde{C}_\beta{}^a{}_b = \widetilde{C}_j{}^i{}_k B_i{}^a B^j{}_\beta N^k{}_b$ and

$$(1.8)\quad H^b_\gamma = N^b_i\,(B_0{}^i{}_\gamma + \Gamma^i{}_k B^k{}_\gamma).$$

The normal curvature vector in a direction $N^i{}_b$ is given by (1.8), while the second fundamental tensor in the same direction is given by

$$(1.9)\quad H_\beta{}^b{}_\gamma = N^b_i\,(B_\beta{}^i{}_\gamma + \Gamma_j{}^i{}_k B^{jk}_{\beta\gamma}) + \widetilde{C}_\beta{}^b{}_c H^c_\gamma,\quad \widetilde{C}_\beta{}^b{}_c = \widetilde{C}_j{}^i{}_k B^j{}_\beta N^b_i N^k_c\,.$$

The h-curvature tensor $R_{\alpha\delta\beta\gamma}$ with respect to $IM\Gamma$ is given by

$$(1.10)\quad \begin{aligned} R_{\alpha\delta\beta\gamma} &= R_{jikh} B^{ji}_{\alpha\delta} B^{kh}_{\beta\gamma} + B^{ji}_{\alpha\delta}\,\{P_{jikh}(B^k{}_\beta H^b_\gamma - B^k{}_\gamma H^b_\beta)\,N^h_b \\ &+ S_{jikh} N^k_b N^h_c H^b_\beta H^c_\gamma\} + [H_\alpha{}^b{}_\beta(g_{jk|\gamma} B^j{}_\delta N^k_b + \delta_{bc} H_\delta{}^c{}_\gamma) - \beta\,|\,\gamma\,], \end{aligned}$$

where R_{jikh}, P_{jikh}, S_{jikh} are the h-, hv-, v- curvature tensors with respect to $M\Gamma$ respectively and the symbol $\beta\,|\,\gamma$ means the interchange of indices β and γ in the foregoing terms.

Contracting (1.10) by $y^\alpha y^\beta$, we have

$$(1.11)\quad \begin{aligned} R_{0\delta 0\gamma} &= R_{0i0h} B^{ih}_{\delta\gamma} + S_{0ikh} B^i{}_\delta N^k_a N^h_b H^a_0 H^b_\gamma \\ &+ B^i{}_\delta N^h_a\,(P_{0i0h} H^a_\gamma - P_{0ikh} B^k{}_\gamma H^a_0) + H_0{}^a{}_0(g_{jk|\gamma} B^j{}_\delta N^k_a + \delta_{ab} H_\delta{}^b{}_\gamma) \\ &- H_0{}^a{}_\gamma\,(g_{jk|\beta}\, y^\beta B^j{}_\delta N^k_a + \delta_{ab} H_\delta{}^b{}_0). \end{aligned}$$

§2. Totally geodesic subspaces. We shall firstly consider Theorem A. An n-dimensional Riemannian space of constant curvature R is generalized to the following two cases:

1) M_n is of constant curvature R with respect to $M\Gamma$ i.e.,

$$(2.1)\quad R_{ijkh}\, y^i y^k = R\,L^2\, h_{jh}\quad (R\text{: constant}),$$

where h_{jh} is the angular metric tensor.

2) M_n is h-isotropic with R with respect to $M\Gamma$ i.e.,

(2.2) $R_{ijkh} = R(g_{ik}g_{jh} - g_{ih}g_{jk})$.

Note 2.1. In the cace 2), R is not always constant. Contracting (2.2) by $y^i y^k$, we have $R_{ojok} = RL^2 h_{jh}$. Therefore M_n is always of scalar curvature R.

A subspace M_m is said to be totally auto-parallel with respect to $IM\Gamma$ if each path in M_m with respect to $IM\Gamma$ is always a path in M_n with respect to $M\Gamma$. And it is known [5] that M_m is totally auto-parallel if and only if the following holds:

(2.3) $H^a_\gamma = 0 \quad (a = m+1, \ldots\ldots, n)$.

And further it follows from (2.3) that

(2.4) $H_\beta{}^a{}_\gamma = Q_\beta{}^a{}_\gamma \; (:= Q_j{}^i{}_k N^a_i B^{jk}_{\beta\gamma})$.

Contracting (2.4) by y^β, we have

(2.5) $H_o{}^a{}_\gamma \, (= y^\beta H_\beta{}^a{}_\gamma) = Q_o{}^a{}_\gamma = D^a{}_\gamma \; (= D^i{}_k N^a_i B^k{}_\gamma)$,

where $D^i{}_k (= \Gamma_j{}^i{}_k y^j - \Gamma^i{}_k = Q_o{}^i{}_k)$ is the deflexion tensor on M_n.

Now we shall impose an assumption on $IM\Gamma$

(2.6) $D^a{}_\gamma = 0 \quad (a = m+1, \ldots\ldots\ldots, n)$,

which is called the D-condition.

Note 2.2. If an $M\Gamma$ satisfies any one of the following axioms then the induced connection $IM\Gamma$ satisfies the D-condition:

(F2) $M\Gamma$ is dft-free ($D^i{}_k = 0$).

(F2)$_1$ $M\Gamma$ is dft-angular ($D^i{}_k = f(x, y) L h^i{}_k$),

where $f(x, y)$ is a $(0)p$-homogeneous scalar and $h^i{}_k = g^{ij} h_{jk}$.

First we can state

Theorem 2.1. *Suppose that a Finsler space M_n is of constant curvature R with respect to $M\Gamma$ and a subspace M_m of M_n is totally auto-parallel with respect to $IM\Gamma$. Then if $IM\Gamma$ satisfies the D-condition, then M_m is of constant curvature R with respect to $IM\Gamma$.*

Proof. Applying (2.1), (2.3), (2.5) and (2.6) to (1.11), we obtain $R_{o\delta o\gamma} = R\bar{L}^2 h_{\delta\gamma}$, where $h_{\delta\gamma} = h_{ij} B^{ij}_{\delta\gamma}$. Q.E.D.

An $M\Gamma$ is called a geo-path connection if any path in M_n with respect to $M\Gamma$ is always a geodesic in M_n. In this case, the induced connection $IM\Gamma$ is also a geo-path connection on M_m [5]. Consequently we can state

Theorem 2. 2. *Suppose that M_n is endowed with a geo-path connection $M\Gamma$ and the induced connection $IM\Gamma$ satisfies the D-condition. Then if M_n is of constant curvature R with respect to $M\Gamma$ and M_m is totally geodesic, then M_m is of constant curvature R with respect to $IM\Gamma$.*

Note 2. 3. The range of validity of the above theorem is extensive.

(I) All the TM(or TM(o))-connections, which are characterized by the following axioms (F1), (F2), (F3), (F4) and (F5) (or $(F5)_o$):

(F1) $M\Gamma$ is metrical ($L_{|k} = 0$). (F2) $M\Gamma$ is dft-free.

(F3) $M\Gamma$ is a geo-path connection.

(F4) $M\Gamma$ satisfies $y^i Dg_{ij} = 0$, where Dg_{ij} is the absolute differential of g_{ij}.

(F5) $M\Gamma$ is v-metrical and v-symmetric ($\widetilde{C}_j{}^i{}_k = C_j{}^i{}_k$).

$(F5)_o$ $M\Gamma$ is v-natural ($\widetilde{C}_j{}^i{}_k = 0$).

The Cartan connection $C\Gamma = (\Gamma^{*}{}_j{}^i{}_k, G^i{}_k, C_j{}^i{}_k)$ and the Hashiguchi connection $H\Gamma = (G_j{}^i{}_k, G^i{}_k, C_j{}^i{}_k)$ are the special TM-connections, while the Rund connection $R\Gamma = (\Gamma^{*}{}_j{}^i{}_k, G^i{}_k, 0)$ and the Berwald connection $B\Gamma = (G_j{}^i{}_k, G^i{}_k, 0)$ are the special TM(o)-connections.

(II) All the TMD(or TMD(o))-connections satisfying the axiom$(F2)_I$, where an TMD(or TMD(o))-connection is characterized by the four axioms (F1), (F3), (F4) and (F5) (or $(F5)_o$). We shall give their examples.

(1) $AMD\Gamma$(or $AMD\Gamma_o$): $\Gamma^i{}_k = G^i{}_k - fLh^i{}_k$, $\Gamma_j{}^i{}_k = \Gamma^{*}{}_j{}^i{}_k + fLC_j{}^i{}_k$.

The connections (1) further satisfy the following desirable axioms:

(F6) $M\Gamma$ is h-symmetric ($\Gamma_j{}^i{}_k = \Gamma_k{}^i{}_j$).

(F7) $M\Gamma$ is h-metrical ($g_{ij|k} = 0$).

(2) $MD\Gamma$(or $MD\Gamma_o$): $\Gamma^i{}_k = G^i{}_k$, $\Gamma_j{}^i{}_k = \Gamma^{*}{}_j{}^i{}_k + f(l_j \delta^i{}_k - l^i g_{jk})$,
where $l_j = L_{\|j}$ and $l^i = y^i/L$.

(3) $AMBD\Gamma$ (or $AMBD\Gamma_o$): $\Gamma^i{}_k = G^i{}_k$, $\Gamma_j{}^i{}_k = G_j{}^i{}_k + f(l_j h^i{}_k + l_k h^i{}_j - l^i h_{jk})$.

(4) $AMCD\Gamma$ (or $AMCD\Gamma_o$): $\Gamma^i{}_k = G^i{}_k$, $\Gamma_j{}^i{}_k = \Gamma^{*}{}_j{}^i{}_k + f(l_j h^i{}_k + l_k h^i{}_j - l^i h_{jk})$.

Further the connections (3) and (4) satisfy (F6), while the connections (2) satisfy (F7) and the following axiom (F8):

(F8) $M\Gamma$ is h-semi-symmetric ($\Gamma_j{}^i{}_k - \Gamma_k{}^i{}_j = \delta^i{}_j s_k - \delta^i{}_k s_j$, being $s_j = -fl_j$).

Note 2. 4. If M_m is totally geodesic then the induced connection $IM\Gamma$ becomes the intrinsic connection on M_m (if it can be defined).

Next we shall consider the case 2). From Note 2. 1 we can state

Theorem 2. 3. *Suppose that M_n is endowed with a geo-path connection $M\Gamma$ and the induced connection $IM\Gamma$ satisfy the D-codition. Then if M_n is h-isotropic with R with respect to $M\Gamma$ and M_m is totally geodesic then M_m is of scalar curvature R with respect to $IM\Gamma$.*

For the special cases, it is known [4] that if $M_n (n \geqq 3)$ is h-isotropic with R with respect to $M\Gamma$ then the following facts hold:

(1) When $M\Gamma = B\Gamma$, R is constant.

(2) When $M\Gamma = R\Gamma$, R is constant together with $H_j{}^i{}_{kh} = K_j{}^i{}_{kh}$,
where $H_j{}^i{}_{kh}$ and $K_j{}^i{}_{kh}$ are the h-curvature tensors with respect to $B\Gamma$ and $R\Gamma$.

(3) When $M\Gamma = H\Gamma$, $R = 0$ or M_n is a Riemannian space of constant curvature R.

(4) When $M\Gamma = C\Gamma$, R is constant together with $S_j{}^i{}_{kh} = 0$,
where $S_j{}^i{}_{kh}$ is the h-curvature tensor with respect to $C\Gamma$.

For $B\Gamma$ and $H\Gamma$ we have $Q_j{}^i{}_k = 0$, while for $R\Gamma$ and $C\Gamma$ we get $Q_j{}^i{}_k = -P_j{}^i{}_k$, where $P_j{}^i{}_k$ is the hv-torsion tensor with respect to $C\Gamma$. Consequently from (1. 10), (1. 11) and (2. 1) ~ (2. 5) we can state

Corollary 2. 3. 1. *If $M_n (n \geqq 3)$ is h-isotropic with R with respect to $M\Gamma$ ((1)$B\Gamma$, (2)$R\Gamma$, (3)$H\Gamma$ and (4)$C\Gamma$) and M_m is totally geodesic then the following facts hold:*

(1) *M_m is h-isotropic with the constant R with respect to $IB\Gamma$.*

(2) *M_m is of constant curvature R with respect to $IR\Gamma$.*

(3) *M_m is h-flat($R_{\alpha\beta\gamma\delta} = 0$) with respect to $IH\Gamma$ or M_m is a Riemannian space of constant curvature R.*

(4) *M_m is of constant curvature R with respect to $IC\Gamma$.*

Note 2. 5. The induced connections in Corollary 2. 3. 1 are all the intrinsic connections on M_m. It seems to us that the case (1) is the most natural generalization of Theorem A.

§3. Totally ncd-free (or nc-constant) subspaces. We shall secondly consider Theorem B. We shall call a $TM\Gamma$ and a $TMD\Gamma$ (resp. a $TM\Gamma_o$ and a $TMD\Gamma_o$) a T-connection (resp. a T(o)-connection) generically and denote it by $T\Gamma$ (resp. $T\Gamma_o$). In the following, we assume that M_n is endowed with a $T\Gamma$ (or $T\Gamma_o$) and M_m is endowed with the induced connection $IT\Gamma$ (or $IT\Gamma_o$). We put

$$(3.1)\qquad \overset{b}{H}{}_{\beta}{}^{a}{}_{\gamma} = N_i^a(B_{\beta}{}^i{}_{\gamma} + G_j{}^i{}_k B^{jk}_{\beta\gamma}),\ \ \overset{b}{H}{}^a_{\gamma} = y^{\beta} H_{\beta}{}^a{}_{\gamma} = N_i^a(B_o{}^i{}_{\gamma} + G^i{}_k B^k{}_{\gamma}),$$

$$(3.2)\qquad \overset{b}{H}{}^a_o = \overset{b}{H}{}^a_{\gamma} y^{\gamma} = N_i^a(B_o{}^i{}_o + 2G^i),\ \ B_o{}^i{}_o = B_o{}^i{}_{\gamma} y^{\gamma},\ \ 2G^i = G^i{}_k y^k.$$

Since the axiom (F3) means $T^i{}_o(=T^i{}_k y^k) = 0$, from (1.1), (1.8) and (3.2) we have

$$(3.3)\qquad H^a_o\ (= H^a_{\gamma} y^{\gamma}) = \overset{b}{H}{}^a_o.$$

Let $f(u^{\alpha}, y^{\alpha})$ be a scalar on M_m. Then we shall say that f is *direct-free* if it is independent of y^{γ} i. e., $f_{\parallel\gamma} = 0$.

We shall call a point (u^{α}) of M_m an *ncd-free* (resp. *nc-constant*) *point* if the following relation holds at the point (u^{α}) for direct-free scalars f^a (resp. constants f^a):

$$(3.4)\qquad \overset{b}{H}{}^a_o = \bar{L}^2 f^a\ (a = m+1, \ldots, n).$$

We shall say that M_m is *totally ncd-free* (resp. *nc-constant*) if every point of M_m is an ncd-free (resp. nc-constant) point.

Note 3.1. An ncd-free (resp. nc-constant) point corresponds to an umbilical point (resp. a proper umbilical point) in Riemannian geometry.

Now we take a curve C: $u^{\alpha} = u^{\alpha}(s)$ (s: arc-length) in M_m. If $du^{\alpha} = y^{\alpha}(ds/\bar{L})$ then because of (3.3) we obtain

$$H^a_{\gamma}(u^{\alpha}, du^{\alpha}/ds)\, du^{\gamma}/ds = \overset{b}{H}{}^a_o(u^{\alpha}, du^{\alpha})/\bar{L}^2(u^{\alpha}, du^{\alpha})$$
$$= \overset{b}{H}{}^a_o(u^{\alpha}, y^{\alpha})/\bar{L}^2(u^{\alpha}, y^{\alpha}),$$

which implies together with (3.4) that the square of normal curvature $N(u^{\alpha}, y^{\alpha})$ in y^{α}-direction is given by $N^2 = \delta_{ab} f^a f^b$.

Note 3.2. If the induced connection $IT\Gamma$ (or $IT\Gamma_o$) satisfies the D-condition then we have

$$H_\gamma^a\, du^\gamma/ds = H_\beta{}^a{}_\gamma\, du^\beta du^\gamma / g_{\beta\gamma} du^\beta du^\gamma = \overset{b}{H}{}_o^a / \bar{L}^2$$

Making use of (1. 6) and (3. 1) ~ (3. 4) we can deduce the following two equivalent equations :

$$\text{(3. 5)} \qquad \overset{b}{H}{}_\gamma^a = f^a y_\gamma - \frac{1}{2} \bar{L}^2 \lambda_{b\gamma}^a f^b, \quad y_\gamma = g_{\beta\gamma} y^\beta,$$

$$\text{(3. 6)} \qquad \overset{b}{H}{}_\beta{}^a{}_\gamma = f^a g_{\beta\gamma} - f^b(\lambda_{b\beta}^a y_\gamma + \lambda_{b\gamma}^a y_\beta) - \frac{1}{2} \bar{L}^2 f^b (\lambda_{b\gamma\|\beta}^a - \lambda_{c\beta}^a \lambda_{b\gamma}^c).$$

Thus we can state

Proposition 3. 1. *Let M_n be endowed with a $T\Gamma$ (or $T\Gamma_o$). Then M_m is totally ncd-free (resp. nc-constant) if and only if any one of the following facts holds for direct-free scalars f^a (resp. constants f^a) :*

(1) *Each normal curvature vector with respect to $IB\Gamma$ is expressed in* (3. 5).

(2) *Each second fundamental tensor with respect to $IB\Gamma$ is expressed in* (3. 6).

For the sake of brevity, we impose another assumption on $IT\Gamma$ (or $IT\Gamma_o$)

$$\text{(3. 7)} \qquad T^a{}_\gamma (= T^i{}_k N_i^a B^k{}_\gamma) = 0, \quad D^a{}_\gamma = 0, \quad Q_\beta{}^a{}_o (= Q_\beta{}^a{}_\gamma y^\gamma) = 0,$$

which is called the *TDQ*-condition.

Note 3. 3. For a $T\Gamma$ (or $T\Gamma_o$) such that $T^i{}_k = 0$ or $T^i{}_k = fLh^i{}_k$, $D^i{}_k = 0$ or $D^i{}_k = \tilde{f} Lh^i{}_k$ and $Q_j{}^i{}_k = 0$ or $Q_j{}^i{}_o = 0$, the induced connection $IT\Gamma$ (or $IT\Gamma_o$) satisfies the *TDQ*-condition. See Lemma 3. 1 in [6].

On making use of Proposition 3. 1 and the TDQ-condition (in fact (1. 1) and (3. 4) ~ (3. 7)), we can obtain [6]

$$\text{(3. 8)} \qquad \begin{aligned} &\frac{1}{2}(R_{o\gamma o\delta} + R_{o\delta o\gamma}) = \frac{1}{2}(R_{oioh} + R_{ohoi}) B^i{}_\gamma B^h{}_\delta + \bar{L}^2 N^2 h_{\gamma\delta} + \frac{1}{2}\Phi_{\gamma\delta}, \\ &\Phi_{\gamma\delta} = U_{\gamma\delta} + V_{\gamma\delta}, \end{aligned}$$

$$\text{(3. 9)} \qquad \begin{aligned} U_{\gamma\delta} = \bar{L}^2 \{ &\bar{L}^2 (f_a \lambda_{b\gamma}^a \lambda_{c\delta}^b f^c - C_{ab\delta\|\gamma} f^a f^b - \frac{1}{2} \sum_a \lambda_{b\gamma}^a \lambda_{c\delta}^a f^b f^c) \\ &- (C_{ab\gamma} y_\delta + C_{ab\gamma\delta} y_\gamma) f^a f^b \}, \text{ being } f_a = \delta_{ab} f^b, \end{aligned}$$

$$\text{(3. 10)} \qquad \begin{aligned} V_{\gamma\delta} = \bar{L}^2 \{ &f_a (Q_\gamma{}^a{}_\delta + Q_\delta{}^a{}_\gamma) + (g_{jk|h} - P_{ojkh}) N_a^k f^a (B_{\gamma\delta}^{ih} + B_{\delta\gamma}^{ih}) - \\ &(g_{jk|h} y^h - P_{ojok}) N_a^k \{ f^a (B^j{}_\gamma y_\delta + B^j{}_\delta y_\gamma) - \frac{1}{2} \bar{L}^2 (B^j{}_\gamma \lambda_{b\delta}^a + B^j{}_\delta \lambda_{b\gamma}^a) f^b \}. \end{aligned}$$

We put

$$T_{ijk}=g_{js}T_i{}^s{}_k,\quad Q_{ijk}=g_{js}Q_i{}^s{}_k,\quad T_{ij}=g_{is}T^s{}_j,\quad T_{\alpha\gamma\delta}=T_{ijk}N^i_\alpha B^{jk}_{\gamma\delta},$$

(3.11) $$T_{\gamma a}=T_{ij}B^j{}_\gamma N^i_a,\quad T^\varepsilon{}_\delta=T^i{}_kB^k{}_\delta B_i{}^\varepsilon,\quad D^\varepsilon{}_\delta=D^i{}_kB^k{}_\delta B_i{}^\varepsilon,$$

$$C_{\varepsilon\gamma a}=C_{\varepsilon a\gamma}=C_{ijk}B^{ij}_{\varepsilon\gamma}N^k_a,\quad P_{\gamma a\delta}=P_{jik}N^i_aB^{jk}_{\gamma\delta},\quad P_{jik}=g_{is}P_j{}^s{}_k.$$

By the use of (3.11) we obtain

(3.12)
$$g_{ij|k}=-T_{ijk}-T_{jik}-Q_{ijk}-Q_{jik}-2(C_{ijr}T^r{}_k+P_{ijk}).$$
$$g_{ij|k}y^k=T_{ij}+T_{ji}-Q_{ijo}-Q_{jio}.$$

The contracted *hv*-curvature tensors P_{oikh} and P_{oioh} are given by

(3.13)
$$P_{oikh}=g_{si}(-Q_h{}^s{}_k+D^s{}_{k\|h})+[C_{jih}D^j{}_k],$$
$$P_{oioh}=g_{si}(-Q_h{}^s{}_o+D^s{}_{o\|h}-D^s{}_h)+[C_{jih}D^j{}_o],$$

where the terms in blackets vanish for $T\Gamma_o$.

Because of $T^a{}_\gamma=D^a{}_\gamma=0$, we get

(3.14) $$C_{j\gamma a}T^j{}_\delta=C_{\varepsilon\gamma a}T^\varepsilon{}_\delta,\quad C_{j\gamma a}D^j{}_\delta=C_{\varepsilon\gamma a}D^\varepsilon{}_\delta,$$

where $C_{j\gamma a}=C_{jik}B^i{}_\gamma N^k_a$, $T^j{}_\delta=T^j{}_kB^k{}_\delta$ and $D^j{}_\delta=D^j{}_kB^k{}_\delta$.

Applying (3.11)～(3.14) to (3.10), we obtain

(3.15)
$$V_{\gamma\delta}=-\bar{L}^2\{T_{a\gamma\delta}+T_{a\delta\gamma}+2(C_{\gamma a\varepsilon}T^\varepsilon{}_\delta+C_{\delta a\varepsilon}T^\varepsilon{}_\gamma+2P_{\gamma a\delta})$$
$$+(g_{s\gamma}B^h{}_\delta+g_{s\delta}B^h{}_\gamma)D^s{}_{h\|k}N^k_a\}f^a+\{\tfrac{1}{2}\bar{L}^2(T_{\gamma a}\lambda^a_{b\delta}+T_{\delta a}\lambda^a_{b\gamma})$$
$$-(T_{\gamma b}y_\delta+T_{\delta b}y_\gamma)\}f^b+g_{sj}(D^s{}_{o\|k}-D^s{}_k)N^k_a\{f^a(B^j{}_\delta y_\gamma+B^j{}_\gamma y_\delta)$$
$$-\tfrac{1}{2}\bar{L}^2(B^j{}_\gamma\lambda^a_{b\delta}+B^j{}_\delta\lambda^a_{b\gamma})f^b\}+[-\bar{L}^2(C_{\gamma a\varepsilon}D^\varepsilon{}_\delta+C_{\delta a\varepsilon}D^\varepsilon{}_\gamma)f^a$$
$$+\{C_{\varepsilon\gamma a}y_\delta+C_{\varepsilon\delta a}y_\gamma-\tfrac{1}{2}\bar{L}^2(C_{\varepsilon\gamma b}\lambda^b_{a\delta}+C_{\varepsilon\delta b}\lambda^b_{a\gamma})\}D^\varepsilon{}_of^a],$$

where all the terms in blackets vanish for $IT\Gamma_o$.

Now we assume that the tensor $\Phi_{\gamma\delta}$ formd with (3.9) and (3.15) may be expressed in

(3.16) $$\Phi_{\gamma\delta}=2\bar{L}^2\mu h_{\gamma\delta}.$$

When (2.1) is valid, from (3.8) and (3.16) we have

(3.17) $\frac{1}{2}(R_{o\gamma o\delta} + R_{o\delta o\gamma}) = \bar{L}^2(R + N^2 + \mu)\, h_{\gamma\delta}.$

Hence we can state

Theorem 3.2. *Suppose that M_n is endowed with a $T\Gamma$ (or $T\Gamma_o$) and the induced connection $IT\Gamma$ (or $IT\Gamma_o$) satisfies the TDQ-condition and that the tensor $\Phi_{\gamma\delta}$ is expressible in the form* (3.16) *for a scalar μ (resp. a constant μ). Then if M_n is of scalar curvature R (resp. of constant curvature R) with respect to $T\Gamma$ (or $T\Gamma_o$) and M_m is totally ncd-free (resp. nc-constant) with N then M_m is of scalar curvature $(R + N^2 + \mu)$ (resp. constant curvature $(R + N^2 + \mu)$) with respect to $IT\Gamma$ (or $IT\Gamma_o$).*

Note 3.4. The above theorem is comparatively complicated because it contains a number of indefinite parts. In (3.16), the scalar μ may be zero. In this case, the tensor $\Phi_{\gamma\delta}$ has to vanish.

Now we require a new assumption

(2.8) $g_{ij}N^i_a N^j_{b\|\gamma} = g_{ij}N^i_b N^j_{a\|\gamma}$ i.e., $\lambda^a_{b\gamma} = \lambda^b_{a\gamma}$,

which will be called the *commutative normality condition* (simply *CN-condition*). In this case, from (1.6), (3.9) and (3.18) we get

(3.19) $U_{\gamma\delta} = \bar{L}^2\{\bar{L}^2(\frac{1}{2}C_{ac\gamma}C_b{}^c{}_\delta - C_{ab\delta\|\gamma}) - (C_{ab\gamma}y_\delta + C_{ab\delta}y_\gamma)\}f^a f^b,$

where $C_{ab\gamma} = C_{ijk}N^i_a N^j_b B^k{}_\gamma = C_b{}^a{}_\gamma = C_a{}^b{}_\gamma.$

In the following, we shall consider only a special case where $T^i{}_k = D^i{}_k = 0$. In this case, (3.15) reduces to

(3.20) $V_{\gamma\delta} = -4\bar{L}^2 P_{\gamma a\delta} f^a.$

Applying (3.9) and (3.20) to (3.16), we obtain

(3.21) $2\mu h_{\gamma\delta} + 4P_{\gamma a\delta}f^a + \bar{L}^2(C_{ab\delta\|\gamma} + C_{ab\gamma}y_\delta + C_{ab\delta}y_\gamma - \frac{1}{2}C_{ac\gamma}C_b{}^c{}_\delta)f^a f^b = 0.$

Further we require another assumption which will be called the *special normality condition* (simply *SN-condition*)

(3.22) $C_{ab\gamma}\, f^a = 0,$

which means $g_{ij}N^i_b \bar{N}^j{}_{\|\gamma} = 0$, being $\bar{N}^j = N^j_a f^a$

Under the *SN*-condition, (3.21) reduces to

(3.23) $\mu h_{\gamma\delta} + 2P_{\gamma\alpha\delta}f^{\alpha} = 0.$

Thus we can state

Corollary 3.2.1. *Suppose that M_n is endowed with a $TM\Gamma$ (or $TM\Gamma_o$) whose non-linear connection is given by $\Gamma^i{}_k = G^i{}_k$ and the induced connection $ITM\Gamma$ (or $ITM\Gamma_o$) satisfies $Q_\beta{}^\alpha{}_o = 0$. Then if M_n is of scalar curvature R (resp. of constant curvature R) with respect to $TM\Gamma$ (or $TM\Gamma_o$) and M_m is totally ncd-free (resp. nc-constant) with N then M_m is of scalar curvature $(R + N^2 + \mu)$ (resp. of constant curvature $(R + N^2 + \mu)$) with respect to $ITM\Gamma$ (or $ITM\Gamma_o$) under the following facts :*

(1) *For a scalar μ (resp. a constant μ), the relation* (3.21) *holds under the CN-condition.*

(2) *For a scalar μ (resp. a constant μ), the relation* (3.23) *holds under both the CN- and SN-conditions.*

Note 3.5. The range of validity of the above corollary is comparatively broad. Examples are as follows :

(I) All the typical connections $B\Gamma, R\Gamma, H\Gamma$ and $C\Gamma$.

(II) All connections with the following form :

$$\Gamma = (\Gamma_j{}^i{}_k, G^i{}_k, C_j{}^i{}_k (\text{or } 0)), \quad \Gamma_j{}^i{}_k = G_j{}^i{}_k + Q_j{}^i{}_k,$$

$$Q_j{}^i{}_k = Z_j{}^i{}_k + f(x, y) h^i{}_j l_k,$$

where $Z_j{}^i{}_k$ is any (o)p-homogeneous tensor satisfying $Z_o{}^i{}_k = Z_j{}^i{}_o = Z_j{}^o{}_k = 0$.

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