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On Conformally Flat Tangent Bundles

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ABSTRACT. Let M be an n-dimensional Riemannian manifold and TM its tangent bundle with the metric I+II or II+III. The tangent bundle TM is conformally flat if and only if M is locally Euclidean.

1. Introduction

In the present paper everything will be always discussed in the c^{∞} category, and Riemannian manifolds will be assumed to be connected and dimension > 1. Let M be an n-dimensional Riemannian manifold with metric g and TM its tangent bundle.It is well known that M with the constant curvature is a conformally flat space, and in the previous paper([1]), we proved that TM with the complete lift metric is conformally flat if and only if M is a space of constant curvature ($n > 2$). The purpose of the present paper is to prove the following.

Theorem. *Let M be an n-dimensional Riemannian manifold and TM its tangent bundle with the metric I+II or II+III. The tangent bundle TM is conformally flat if and only if M is locally Euclidean.*

2. Pleliminaries

Let $\{X_h, X_{\overline{h}}\}$ be the adapted frame of TM :

$$X_h = \frac{\partial}{\partial x^h} - y^a \Gamma_a{}^b{}_h \frac{\partial}{\partial y^b} \qquad \text{and} \qquad X_{\overline{h}} = \frac{\partial}{\partial y^h},$$

and let $\{dx^h, \delta y^h\}$ be the dual basis of $\{X_h, X_{\overline{h}}\}$, where$(x^h, y^h)$ are the induced coordinates in TM and $\Gamma_a{}^b{}_h$ the components of the Riemannian connection of M. The indices $a,b,c,...,h,i,j,...,$ run over the range $\{1,2,...,n\}$. The indices $A,B,C,...,P,Q,R,...,$run over the range $\{1,2,...,n,\overline{1},\overline{2},...,\overline{n}\}$ and the indices $\overline{a},\overline{b},\overline{c},...,\overline{h},\overline{i},\overline{j},...,$ run over the range $\{\overline{1},\overline{2},...,\overline{n}\}$.The summation convention will be used in relation to this system of indices. By the straightforward calculations, we have the following lemma.

Lemma 1. *The Lie brackets of the adapted frame of TM satisfy the following :*

$$(1) \quad [X_i, X_j] = y^a K_{jia}{}^m X_{\overline{m}},$$

$$(2) \quad [X_i, X_{\overline{j}}] = \Gamma_j{}^m{}_i X_{\overline{m}},$$

$$(3) \quad [X_{\overline{i}}, X_{\overline{j}}] = 0,$$

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where $K_{jia}{}^{m}$ *denote the components of the curvature tensor of* M.

Let $\overline{\nabla}$ be the Riemannian connection of TM and $\overline{\Gamma}_{B}{}^{A}{}_{C}$ the coefficients of $\overline{\nabla}$:

$$\text{(2-1)}\quad \begin{aligned} \overline{\nabla}_{X_i} X_j &= \overline{\Gamma}_{j}{}^{m}{}_{i} X_m + \overline{\Gamma}_{j}{}^{\overline{m}}{}_{i} X_{\overline{m}}, & \overline{\nabla}_{X_i} X_{\overline{j}} &= \overline{\Gamma}_{\overline{j}}{}^{m}{}_{i} X_m + \overline{\Gamma}_{\overline{j}}{}^{\overline{m}}{}_{i} X_{\overline{m}}, \\ \overline{\nabla}_{X_{\overline{i}}} X_j &= \overline{\Gamma}_{j}{}^{m}{}_{\overline{i}} X_m + \overline{\Gamma}_{j}{}^{\overline{m}}{}_{\overline{i}} X_{\overline{m}}, & \overline{\nabla}_{X_{\overline{i}}} X_{\overline{j}} &= \overline{\Gamma}_{\overline{j}}{}^{m}{}_{\overline{i}} X_m + \overline{\Gamma}_{\overline{j}}{}^{\overline{m}}{}_{\overline{i}} X_{\overline{m}}. \end{aligned}$$

We denote g^{I+II} the metric $I+II$ and g^{II+III} the metric $II+III$, which are defined as follows :

$$\text{(2-2)}\qquad g^{I+II} = g_{ij} dx^i dx^j + 2g_{ij} dx^i \delta y^j,$$

$$\text{(2-3)}\qquad g^{II+III} = 2g_{ij} dx^i \delta y^j + g_{ij} \delta y^i \delta y^j.$$

We have the following lemmas (see [2]).

Lemma 2. *The connection coefficients* $\overline{\Gamma}_{B}{}^{A}{}_{C}$ *of* g^{I+II} *satisfy the following :*

(1) $\overline{\Gamma}_{j}{}^{h}{}_{i} = \Gamma_{j}{}^{h}{}_{i}$, (2) $\overline{\Gamma}_{j}{}^{\overline{h}}{}_{i} = \frac{1}{2} y^a K_{aij}{}^{h}$, (3) $\overline{\Gamma}_{\overline{j}}{}^{h}{}_{i} = 0$,

(4) $\overline{\Gamma}_{j}{}^{h}{}_{\overline{i}} = 0$, (5) $\overline{\Gamma}_{\overline{j}}{}^{\overline{h}}{}_{i} = \Gamma_{j}{}^{h}{}_{i}$, (6) $\overline{\Gamma}_{j}{}^{\overline{h}}{}_{\overline{i}} = 0$,

(7) $\overline{\Gamma}_{\overline{j}}{}^{h}{}_{\overline{i}} = 0$, (8) $\overline{\Gamma}_{\overline{j}}{}^{\overline{h}}{}_{\overline{i}} = 0$.

Lemma 3. *The connection coefficients* $\overline{\Gamma}_{B}{}^{A}{}_{C}$ *of* g^{II+III} *satisfy the following :*

(1) $\overline{\Gamma}_{j}{}^{h}{}_{i} = \Gamma_{j}{}^{h}{}_{i} - \frac{1}{2} y^a (K_{aij}{}^{h} + K_{aji}{}^{h})$, (2) $\overline{\Gamma}_{j}{}^{\overline{h}}{}_{i} = y^a K_{aij}{}^{h}$,

(3) $\overline{\Gamma}_{\overline{j}}{}^{h}{}_{i} = -\frac{1}{2} y^a K_{aji}{}^{h}$, (4) $\overline{\Gamma}_{j}{}^{h}{}_{\overline{i}} = -\frac{1}{2} y^a K_{aij}{}^{h}$,

(5) $\overline{\Gamma}_{\overline{j}}{}^{\overline{h}}{}_{i} = \Gamma_{j}{}^{h}{}_{i} + \frac{1}{2} K_{aji}{}^{h}$, (6) $\overline{\Gamma}_{j}{}^{\overline{h}}{}_{\overline{i}} = \frac{1}{2} y^a K_{aij}{}^{h}$,

(7) $\overline{\Gamma}_{\overline{j}}{}^{h}{}_{\overline{i}} = 0$, (8) $\overline{\Gamma}_{\overline{j}}{}^{\overline{h}}{}_{\overline{i}} = 0$.

3. Curvature tensors of TM

The curvature tensor $\overline{K}$ of TM is defined by

$$\overline{K}(X,Y)Z = \overline{\nabla}_X \overline{\nabla}_Y Z - \overline{\nabla}_Y \overline{\nabla}_X Z - \overline{\nabla}_{[X,Y]} Z, \qquad for X, Y, Z \varepsilon TTM,$$

then from Lemma 1, Lemma 2 and Lemma 3, we have the following lemmas.

Lemma 4. *The components of the curvature tensor of* g^{I+II} *are given as follows :*

$$\begin{aligned} &(1)\quad \overline{K}(X_i, X_j)X_k &&= K_{ijk}{}^{m} X_m + y^a \{\nabla_i K_{ajk}{}^{m} - \nabla_j K_{aik}{}^{m}\} X_{\overline{m}}, \\ &(2)\quad \overline{K}(X_i, X_j)X_{\overline{k}} &&= K_{ijk}{}^{m} X_{\overline{m}}, \\ &(3)\quad \overline{K}(X_{\overline{i}}, X_j)X_k &&= K_{ijk}{}^{m} X_{\overline{m}}, \\ &(4)\quad \overline{K}(X_{\overline{i}}, X_j)X_{\overline{k}} &&= 0, \\ &(5)\quad \overline{K}(X_{\overline{i}}, X_{\overline{j}})X_k &&= 0, \\ &(6)\quad \overline{K}(X_{\overline{i}}, X_{\overline{j}})X_{\overline{k}} &&= 0. \end{aligned}$$

Lemma 5. *The components of the curvature tensor of g^{II+III} are given as follows :*

$$
\begin{aligned}
(1)\quad \overline{K}(X_i,X_j)X_k &= [K_{ijk}{}^m - \frac{1}{2}y^a(\nabla_i K_{ajk}{}^m + \nabla_i K_{akj}{}^m - \nabla_j K_{aik}{}^m - \nabla_j K_{aki}{}^m)\\
&\quad + \frac{1}{4}y^a y^b(K_{ajk}{}^c K_{bic}{}^m - K_{aik}{}^c K_{bjc}{}^m - K_{ajk}{}^c K_{bci}{}^m + K_{aik}{}^c K_{bcj}{}^m\\
&\quad + K_{akj}{}^c K_{bic}{}^m - K_{aki}{}^c K_{bjc}{}^m + K_{akj}{}^c K_{bci}{}^m - K_{aki}{}^c K_{bcj}{}^m\\
&\quad + 2K_{jia}{}^c K_{bck}{}^m)]X_m\\
&\quad + [y^a(\nabla_i K_{ajk}{}^m - \nabla_j K_{aik}{}^m) - \frac{1}{2}y^a y^b(K_{ajk}{}^c K_{bic}{}^m - K_{aik}{}^c K_{bjc}{}^m\\
&\quad + K_{akj}{}^c K_{bic}{}^m - K_{aki}{}^c K_{bjc}{}^m - K_{ajk}{}^c K_{bci}{}^m\\
&\quad + K_{aik}{}^c K_{bcj}{}^m + K_{jia}{}^c K_{bck}{}^m)]X_{\overline{m}},\\
(2)\quad \overline{K}(X_i,X_j)X_{\overline{k}} &= [-\frac{1}{2}y^a(\nabla_i K_{akj}{}^m - \nabla_j K_{aki}{}^m) + \frac{1}{4}y^a y^b(K_{akj}{}^c K_{bic}{}^m\\
&\quad - K_{aki}{}^c K_{bjc}{}^m)]X_m\\
&\quad + [K_{ijk}{}^m + \frac{1}{2}y^a(\nabla_i K_{akj}{}^m - \nabla_j K_{aki}{}^m)\\
&\quad - \frac{1}{4}y^a y^b K_{akj}{}^c(2K_{bic}{}^m - K_{bci}{}^m)\\
&\quad + \frac{1}{4}y^a y^b K_{aki}{}^c(2K_{bjc}{}^m - K_{bcj}{}^m)]X_{\overline{m}},\\
(3)\quad \overline{K}(X_{\overline{i}},X_j)X_k &= [-\frac{1}{2}(K_{ikj}{}^m + K_{ijk}{}^m) + \frac{1}{2}y^a\nabla_j K_{aik}{}^m + \frac{1}{4}y^a y^b(K_{akj}{}^c K_{bic}{}^m\\
&\quad + K_{ajk}{}^c K_{bic}{}^m - K_{aik}{}^c K_{bjc}{}^m)]X_m\\
&\quad + [K_{ijk}{}^m - \frac{1}{2}y^a\nabla_j K_{aik}{}^m - \frac{1}{4}y^a y^b(K_{akj}{}^c K_{bic}{}^m + K_{ajk}{}^c K_{bic}{}^m\\
&\quad + K_{aik}{}^c K_{bcj}{}^m - 2K_{aik}{}^c K_{bjc}{}^m)]X_{\overline{m}},\\
(4)\quad \overline{K}(X_{\overline{i}},X_j)X_{\overline{k}} &= [-\frac{1}{2}K_{ikj}{}^m + \frac{1}{4}y^a y^b K_{akj}{}^c K_{bic}{}^m]X_m\\
&\quad + [\frac{1}{2}K_{ikj}{}^m - \frac{1}{4}y^a y^b K_{akj}{}^c K_{bic}{}^m]X_{\overline{m}},\\
(5)\quad \overline{K}(X_{\overline{i}},X_{\overline{j}})X_k &= [-K_{ijk}{}^m + \frac{1}{4}y^a y^b(K_{ajk}{}^c K_{bic}{}^m - K_{aik}{}^c K_{bjc}{}^m)]X_m\\
&\quad + [K_{ijk}{}^m - \frac{1}{4}y^a y^b(K_{ajk}{}^c K_{bic}{}^m - K_{aik}{}^c K_{bjc}{}^m)]X_{\overline{m}},\\
(6)\quad \overline{K}(X_{\overline{i}},X_{\overline{j}})X_{\overline{k}} &= 0.
\end{aligned}
$$

Let $\overline{g}_{AB}$ be the components of the metric g^{I+II} or g^{II+III}, and $\overline{K}_{ABCD}$ the components of the curvature tensor $\overline{K}$ of TM : $\overline{K}_{ABCD} = \overline{g}(\overline{K}(X_A,X_B)X_C,X_D)$. The scalar curvature $\overline{S}$ of TM is defiend by $\overline{S} = \overline{g}^{AD}\overline{g}^{BC}\overline{K}_{ABCD}$, where $\overline{g}^{AB}$ denote the components of the inverse matrix of $(\overline{g}_{AB})$, and the components $\overline{K}_{BC}$ of the Ricci tensor of TM are defined by $\overline{K}_{BC} = \overline{g}^{AD}\overline{K}_{ABCD}$. By means of Lemma 4 and Lemma 5, we have the following lemmas.

Lemma 6([3],[2]). *The scalar curvature $\overline{S}^{I+II}$ of g^{I+II} and the scalar curvature $\overline{S}^{II+III}$ of g^{II+III} are given as follows :*

$$(1)\quad \overline{S}^{I+II} = 0,$$

$$(2)\quad \overline{S}^{II+II} = -S - \frac{1}{2}y^a y^b K_{acde}K_b{}^{cde},$$

where S denotes the scalar curvature of M.

Lemma 7. *The components $\overline{K}_{BC}$ of the Ricci tensor of g^{I+II} are given as follows :*

$$\begin{aligned}
(1)\quad \overline{K}_{jk} &= 2K_{jk},\\
(2)\quad \overline{K}_{\bar{j}k} &= 0,\\
(3)\quad \overline{K}_{j\bar{k}} &= 0,\\
(4)\quad \overline{K}_{\bar{j}\bar{k}} &= 0.
\end{aligned}$$

Lemma 8. *The components $\overline{K}_{BC}$ of the Ricci tensor of g^{II+III} are given as follows:*

$$\begin{aligned}
(1)\quad \overline{K}_{jk} &= 2K_{jk} - \frac{1}{2}y^a(\nabla_a K_{jk} - \nabla_j K_{ak} + \nabla_a K_{kj} - \nabla_k K_{aj})\\
&\quad + \frac{1}{4}y^a y^b(K_{ack}{}^d K_{bjd}{}^c - K_{akc}{}^d K_{bjd}{}^c - K_{akc}{}^d K_{bdj}{}^c + 2K_{jca}{}^d K_{bdk}{}^c),\\
(2)\quad \overline{K}_{\bar{j}k} &= \frac{1}{2}K_{jk} - \frac{1}{2}y^a(\nabla_a K_{jk} - \nabla_j K_{ak}) - \frac{1}{4}y^a y^b K_{ajc}{}^d K_{bkd}{}^c,\\
(3)\quad \overline{K}_{j\bar{k}} &= \frac{1}{2}K_{kj} - \frac{1}{2}y^a(\nabla_a K_{kj} - \nabla_k K_{aj}) - \frac{1}{4}y^a y^b K_{akc}{}^d K_{bjd}{}^c,\\
(4)\quad \overline{K}_{\bar{j}\bar{k}} &= -\frac{1}{4}y^a y^b K_{akc}{}^d K_{bjd}{}^c.
\end{aligned}$$

4. Proof of Theorem

If the components of the curvature tensor of TM satisfy the following equations :

$$\begin{aligned}
(4\text{-}1)\quad \overline{K}_{ABCD} &= \frac{1}{2(n-1)}(\overline{g}_{AD}\overline{K}_{BC} - \overline{g}_{BD}\overline{K}_{AC} + \overline{g}_{BC}\overline{K}_{AD} - \overline{g}_{AC}\overline{K}_{BD})\\
&\quad - \frac{\overline{S}}{2(2n-1)(n-1)}(\overline{g}_{AD}\overline{g}_{BC} - \overline{g}_{BD}\overline{g}_{AC}),
\end{aligned}$$

then TM is said to be conformally flat.

Proof of Theorem in the case of TM with the metric g^{I+II}. From Lemma 4, we have

$$\begin{aligned}
(4\text{-}2)\quad \overline{K}_{ijkl} &= K_{ijkl} + 2y^a(\nabla_i K_{ajkl} - \nabla_j K_{aikl}),\\
(4\text{-}3)\quad \overline{K}_{ijk\bar{l}} &= K_{ijkl},\\
(4\text{-}4)\quad \overline{K}_{ij\bar{k}\bar{l}} &= 2K_{ijkl},\\
(4\text{-}5)\quad \overline{K}_{\bar{i}jkl} &= 2K_{ijkl},\\
(4\text{-}6)\quad \textit{otherwise} &= 0.
\end{aligned}$$

From (4-1), we obtain

$$\begin{aligned}
(4\text{-}7)\quad \overline{K}_{ijkl} &= \frac{1}{2(n-1)}(\overline{g}_{il}\overline{K}_{jk} - \overline{g}_{jl}\overline{K}_{ik} + \overline{g}_{jk}\overline{K}_{il} - \overline{g}_{ik}\overline{K}_{jl})\\
&= \frac{1}{n-1}(g_{il}K_{jk} - g_{jl}K_{ik} + g_{jk}K_{il} - g_{ik}K_{jl}),
\end{aligned}$$

$$(4\text{-}8)\quad \overline{K}_{ijk\bar{l}} = \frac{1}{2(n-1)}(\overline{g}_{i\bar{l}}\overline{K}_{jk} - \overline{g}_{j\bar{l}}\overline{K}_{ik} + \overline{g}_{jk}\overline{K}_{i\bar{l}} - \overline{g}_{ik}\overline{K}_{j\bar{l}})$$
$$= \frac{1}{n-1}(g_{il}K_{jk} - g_{jl}K_{ik}).$$

By means of (4-2),(4-3),(4-7) and (4-8), we get $K_{ijkl} = 0$.

Proof of Theorem in the case of TM with the metric g^{II+III}. From (4-1), we have

$$(4\text{-}9)\quad \overline{K}_{ijkl} = 0,$$

$$(4\text{-}10)\quad \overline{K}_{ijk\bar{l}} = \frac{1}{2(n-1)}(g_{il}\overline{K}_{jk} - g_{jl}\overline{K}_{i\bar{l}}),$$

$$(4\text{-}11)\quad \overline{K}_{ij\bar{k}l} = \frac{1}{2(n-1)}(g_{jk}\overline{K}_{il} - g_{ik}\overline{K}_{jl}),$$

$$(4\text{-}12)\quad \overline{K}_{ij\overline{kl}} = \frac{1}{2(n-1)}(g_{il}\overline{K}_{j\bar{k}} - g_{jl}\overline{K}_{i\bar{k}} + g_{jk}\overline{K}_{i\bar{l}} - g_{ik}\overline{K}_{j\bar{l}})$$
$$-\frac{\overline{S}}{2(2n-1)(n-1)}(g_{il}g_{jk} - g_{jl}g_{ik}),$$

$$(4\text{-}13)\quad \overline{K}_{\bar{i}jkl} = \frac{1}{2(n-1)}(g_{il}\overline{K}_{jk} - g_{ik}\overline{K}_{jl}),$$

$$(4\text{-}14)\quad \overline{K}_{\bar{i}jk\bar{l}} = \frac{1}{2(n-1)}(g_{il}\overline{K}_{jk} - g_{jl}\overline{K}_{\bar{i}k} - g_{ik}\overline{K}_{j\bar{l}})$$
$$+\frac{\overline{S}}{2(2n-1)(n-1)}g_{jl}g_{ik},$$

$$(4\text{-}15)\quad \overline{K}_{\bar{i}j\bar{k}l} = \frac{1}{2(n-1)}(g_{il}\overline{K}_{j\bar{k}} + g_{jk}\overline{K}_{\bar{i}l} - g_{ik}\overline{K}_{jl})$$
$$-\frac{\overline{S}}{2(2n-1)(n-1)}g_{il}g_{jk},$$

$$(4\text{-}16)\quad \overline{K}_{\bar{i}j\overline{kl}} = \frac{1}{2(n-1)}(g_{il}\overline{K}_{j\bar{k}} - g_{jl}\overline{K}_{\overline{\bar{i}k}} + g_{jk}\overline{K}_{\overline{\bar{i}l}} - g_{ik}\overline{K}_{j\bar{l}})$$
$$-\frac{\overline{S}}{2(2n-1)(n-1)}(g_{il}g_{jk} - g_{jl}g_{ik}),$$

$$(4\text{-}17)\quad \overline{K}_{\overline{\bar{i}j}kl} = \frac{1}{2(n-1)}(g_{il}\overline{K}_{\bar{j}k} - g_{jl}\overline{K}_{\bar{i}k} + g_{jk}\overline{K}_{\bar{i}l} - g_{ik}\overline{K}_{\bar{j}l})$$
$$-\frac{\overline{S}}{2(2n-1)(n-1)}(g_{il}g_{jk} - g_{jl}g_{ik}),$$

$$(4\text{-}18)\quad \overline{K}_{\overline{\bar{i}j}k\bar{l}} = \frac{1}{2(n-1)}(g_{il}\overline{K}_{\bar{j}k} - g_{jl}\overline{K}_{\bar{i}k} + g_{jk}\overline{K}_{\overline{\bar{i}l}} - g_{ik}\overline{K}_{\overline{\bar{j}l}})$$
$$-\frac{\overline{S}}{2(2n-1)(n-1)}(g_{il}g_{jk} - g_{jl}g_{ik}),$$

$$(4\text{-}19)\quad \overline{K}_{\overline{\bar{i}jk}l} = \frac{1}{2(n-1)}(g_{il}\overline{K}_{\overline{\bar{j}k}} - g_{jl}\overline{K}_{\overline{\bar{i}k}} + g_{jk}\overline{K}_{\bar{i}l} - g_{ik}\overline{K}_{\bar{j}l})$$
$$-\frac{\overline{S}}{2(2n-1)(n-1)}(g_{il}g_{jk} - g_{jl}g_{ik}),$$

$$(4\text{-}20)\quad \overline{K}_{\overline{i}\overline{j}\overline{k}\overline{l}} = \frac{1}{2(n-1)}(g_{il}\overline{K}_{\overline{j}\overline{k}} - g_{jl}\overline{K}_{\overline{i}\overline{k}} + g_{jk}\overline{K}_{\overline{i}\overline{l}} - g_{ik}\overline{K}_{\overline{j}\overline{l}}) - \frac{\overline{S}}{2(2n-1)(n-1)}(g_{il}g_{jk} - g_{jl}g_{ik}).$$

From (6) of Lemma 5 and (4-19), we obtain

$$(4\text{-}21)\quad g_{il}\overline{K}_{\overline{j}k} - g_{jl}\overline{K}_{\overline{i}k} + g_{jk}\overline{K}_{\overline{i}l} - g_{ik}\overline{K}_{\overline{j}l} = \frac{\overline{S}}{2n-1}(g_{il}g_{jk} - g_{jl}g_{ik}),$$

and from (6) of Lemma 5 and (4-20), we get

$$(4\text{-}22)\quad g_{il}\overline{K}_{\overline{j}\overline{k}} - g_{jl}\overline{K}_{\overline{i}\overline{k}} + g_{jk}\overline{K}_{\overline{i}\overline{l}} - g_{ik}\overline{K}_{\overline{j}\overline{l}} = \frac{\overline{S}}{2n-1}(g_{il}g_{jk} - g_{jl}g_{ik}),$$

it follows that $\overline{K}_{\overline{i}l} = \overline{K}_{\overline{i}\overline{l}}$. By means of (2) and (4) of Lemma 8, we have

$$(4\text{-}23)\quad K_{jk} = 0,$$

and

$$(4\text{-}24)\quad \overline{K}_{\overline{j}\overline{k}} = -\frac{1}{4}y^a y^b K_{ajc}{}^d K_{bkd}{}^c.$$

From (4-22), we obtain

$$(4\text{-}25)\quad (n-2)\overline{K}_{\overline{j}\overline{k}} + g^{ab}g_{jk}\overline{K}_{\overline{a}\overline{b}} = \frac{(n-1)\overline{S}}{2n-1}g_{jk},$$

it follows that

$$(4\text{-}26)\quad 2(n-1)g^{ab}\overline{K}_{ab} = \frac{n(n-1)}{2n-1}\overline{S}.$$

On the other hand, from (4-24), we get

$$(4\text{-}27)\quad g^{ab}\overline{K}_{ab} = \frac{1}{4}y^a y^b K_{acde}K_b{}^{cde} = -\frac{1}{2}\overline{S}.$$

Thus by (4-26) and (4-27), we have $\overline{S}$=0, then it follows

$$(4\text{-}28)\quad y^a y^b K_{acde}K_b{}^{cde} = 0.$$

This shows $K_{ijkl} = 0$. This completes the proof of Theorem. q.e.d.

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