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On Infinitesimal Projective Transformations of Tangent Bundles with the Metric II+III

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ABSTRACT. Let M be an n-dimensional complete Riemannian manifold and TM its tangent bundle with the metric II+III. If TM admits a non-affine infinitesimal fibre-preserving projective transformation, then M is locally Euclidean.

1. Introduction

In the present paper everything will be always discussed in the C^∞ category, and Riemannian manifolds will be assumed to be connected and dimension>1. Let M be a Riemannian manifold, and let let ϕ be a transformation of M. Then ϕ is called a projective transformation, if it preserves the geodesics, where each geodesic should be confounded with a subset of M by neglecting its affine parameter. Furthermore ϕ is called an affine transformation, if it preserves the Riemannian connection. We then remark that an affine transformation may be characterized as a projective transformation which preserves the affine parameter together with the geodesics. Let V be a vector field on M, and let us consider a local one-parameter group $\{\phi_t\}$ of local transformations of M generated by V. Then V is called an infinitesimal projective transformation, if each ϕ_t is a local projective transformation. By a complete infinitesimal projective transformation we mean an infinitesimal projective transformation which generates a global one-parameter group of projective transformations.

Let TM be the tangent bundle of M and g a Riemannian metric of M. Then there are many Riemannian or psued-Riemannian metrics in TM which are defined by g, for example, Sasaki metric, complete lift metric, Cheeger-Gromoll metric, etc. Lex X be a vector field on TM, and $\{\Phi_t\}$ a local one-parameter group of local transformations of TM generated by X. Then X is called an infinitesimal fibre-preserving projective transformation, if each Φ_t is a local fibre-preserving projective transformation of TM.

The purpose of the present paper is to show the following theorem.

Theorem. *Let M be an n-dimensional complete Riemannian manifold and TM its tangent*

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bundle with the metric II+III. If TM admits a non-affine infinitesimal fibre-preserving projective transformation, then M is locally Euclidean.

2. Preliminaries

Let $\{X_h, X_{\bar{h}}\}$ be the adapted frame of TM:

$$X_h = \frac{\partial}{\partial x^h} - y^a \Gamma_a{}^m{}_h \frac{\partial}{\partial y^m} \qquad \text{and} \quad X_{\bar{h}} = \frac{\partial}{\partial y^h},$$

and let $\{dx^h, \delta y^h\}$ be the dual basis of $\{X_h, X_{\bar{h}}\}$, where (x^h, y^h) are the induced coordinates in TM and $\Gamma_a{}^m{}_h$ are the components of the Riemannian connection of M. By straightforward calculations, we have the following lemma.

Lemma 1. *The Lie brackets of the adapted frame of TM satisfy the following:*

(1) $[X_i, X_j] = y^a K_{jia}{}^m X_{\bar{m}}$,

(2) $[X_i, X_{\bar{j}}] = \Gamma_j{}^m{}_i X_{\bar{m}}$,

(3) $[X_{\bar{i}}, X_{\bar{j}}] = 0$,

where $K_{jia}{}^m$ denote the components of the curvature tensor of M.

Let X be an infinitesimal fibre-preserving transformation of TM and $(v^h, v^{\bar{h}})$ the components of X with respect to the adapted frame $\{X_h, X_{\bar{h}}\}$, then the horizontal components v^h depend only on the variables (x^h) because of X being the fibre-preserving. Thus X naturally induces a vector field V on M with the components (v^h). Let L_X be the Lie derivation with respect to X, then we have the following lemma.

Lemma 2. *The Lie derivatives of the adapted frame and the dual basis are given as follows:*

(1) $L_X X_h = -\partial_h v^m X_m + \{y^a v^b K_{hba}{}^m - v^{\bar{b}} \Gamma_b{}^m{}_h - X_h(v^{\bar{m}})\} X_{\bar{m}}$,

(2) $L_X X_{\bar{h}} = \{v^b \Gamma_b{}^m{}_h - X_{\bar{h}}(v^{\bar{m}})\} X_{\bar{m}}$,

(3) $L_X dx^h = \partial_m v^h dx^m$,

(4) $L_X \delta y^h = -\{y^a v^b K_{mba}{}^h - v^{\bar{b}} \Gamma_b{}^h{}_m - X_m(v^{\bar{h}})\} dx^m - \{v^b \Gamma_b{}^h{}_m - X_{\bar{m}}(v^{\bar{h}})\} \delta y^m$.

3. The Riemannian connection of TM with the metric II+III

Let G be the metric II+III of TM: $G = 2g_{ij} dx^i \delta y^j + g_{ij} \delta y^i \delta y^j$. Let $\overline{\nabla}$ be the Riemannian connection of G and $\overline{\Gamma}_B{}^A{}_C$ the coefficients of $\overline{\nabla}$, that is

$$(3.1) \quad \begin{aligned} &\overline{\nabla}_{X_i} X_j = \overline{\Gamma}_j{}^m{}_i X_m + \overline{\Gamma}_j{}^{\bar{m}}{}_i X_{\bar{m}}, && \overline{\nabla}_{X_i} X_{\bar{j}} = \overline{\Gamma}_{\bar{j}}{}^m{}_i X_m + \overline{\Gamma}_{\bar{j}}{}^{\bar{m}}{}_i X_{\bar{m}}, \\ &\overline{\nabla}_{X_{\bar{i}}} X_j = \overline{\Gamma}_j{}^m{}_{\bar{i}} X_m + \overline{\Gamma}_j{}^{\bar{m}}{}_{\bar{i}} X_{\bar{m}}, && \overline{\nabla}_{X_{\bar{i}}} X_{\bar{j}} = \overline{\Gamma}_{\bar{j}}{}^m{}_{\bar{i}} X_m + \overline{\Gamma}_{\bar{j}}{}^{\bar{m}}{}_{\bar{i}} X_{\bar{m}}. \end{aligned}$$

Then we have the following for the dual basis $\{dx^h, \delta y^h\}$:

$$(3.2) \quad \begin{aligned} &\overline{\nabla}_{X_i} dx^h = -\overline{\Gamma}_m{}^h{}_i dx^m - \overline{\Gamma}_{\bar{m}}{}^h{}_i \delta y^m, && \overline{\nabla}_{X_i} \delta y^h = -\overline{\Gamma}_m{}^{\bar{h}}{}_i dx^m - \overline{\Gamma}_{\bar{m}}{}^{\bar{h}}{}_i \delta y^m, \\ &\overline{\nabla}_{X_{\bar{i}}} dx^h = -\overline{\Gamma}_m{}^h{}_{\bar{i}} dx^m - \overline{\Gamma}_{\bar{m}}{}^h{}_{\bar{i}} \delta y^m, && \overline{\nabla}_{X_{\bar{i}}} \delta y^h = -\overline{\Gamma}_m{}^{\bar{h}}{}_{\bar{i}} dx^m - \overline{\Gamma}_{\bar{m}}{}^{\bar{h}}{}_{\bar{i}} \delta y^m. \end{aligned}$$

Since the torsion tensor of $\overline{\nabla}$ vanishes, we have the following lemma by means of Lemma 1 and (3-1).

Lemma 3. *The connection coefficients $\overline{\Gamma}_B{}^A{}_C$ of $\overline{\nabla}$ satisfy the following:*

(1) $\overline{\Gamma}_j{}^h{}_i = \overline{\Gamma}_i{}^h{}_j$, (2) $\overline{\Gamma}_j{}^{\bar{h}}{}_i = \overline{\Gamma}_i{}^{\bar{h}}{}_j + y^a K_{jia}{}^h$, (3) $\overline{\Gamma}_{\bar{j}}{}^h{}_i = \overline{\Gamma}_i{}^h{}_{\bar{j}}$,

(4) $\overline{\Gamma}_{\bar{j}}{}^{\bar{h}}{}_i = \overline{\Gamma}_i{}^{\bar{h}}{}_{\bar{j}} + \Gamma_j{}^h{}_i$, (5) $\overline{\Gamma}_{\bar{j}}{}^h{}_{\bar{i}} = \overline{\Gamma}_{\bar{i}}{}^h{}_{\bar{j}}$, (6) $\overline{\Gamma}_{\bar{j}}{}^{\bar{h}}{}_{\bar{i}} = \overline{\Gamma}_{\bar{i}}{}^{\bar{h}}{}_{\bar{j}}$.

Furthermore, since the connection $\overline{\nabla}$ is metrical, we have the following proposition.

Proposition. *The Riemannian connection* $\overline{\nabla}$ *of* TM *with the metric* $II+III$ *satisfies the following equations.*

(1) $\overline{\nabla}_{X_i}X_j=\{\Gamma_i{}^m{}_j-\frac{1}{2}y^a(K_{aji}{}^m+K_{aij}{}^m)\}X_m+y^aK_{aij}{}^mX_{\bar{m}}$,

(2) $\overline{\nabla}_{X_i}X_{\bar{j}}=-\frac{1}{2}y^aK_{aji}{}^mX_m+\{\Gamma_i{}^m{}_j+\frac{1}{2}y^aK_{aji}{}^m\}X_{\bar{m}}$,

(3) $\overline{\nabla}_{X_{\bar{i}}}X_j=-\frac{1}{2}y^aK_{aij}{}^mX_m+\frac{1}{2}y^aK_{aij}{}^mX_{\bar{m}}$,

(4) $\overline{\nabla}_{X_{\bar{i}}}X_{\bar{j}}=0$.

Proof. By virtue of (3-2) and the connection $\overline{\nabla}$ is metrical, we have

$$
\begin{aligned}
0&=\overline{\nabla}_{X_m}G\\
&=\overline{\nabla}_{X_m}(2g_{ij}dx^i\delta y^j+g_{ij}\delta y^i\delta y^j)\\
&=2\partial_mg_{ij}dx^i\delta y^j+2g_{ij}(\overline{\nabla}_{X_m}dx^i)\delta y^j+2g_{ij}dx^i(\overline{\nabla}_{X_m}\delta y^j)+\partial_mg_{ij}\delta y^i\delta y^j+2g_{ij}(\overline{\nabla}_{X_m}\delta y^i)\delta y^j\\
&=2\partial_mg_{ij}dx^i\delta y^j+2g_{ij}(-\overline{\Gamma}_r{}^{\bar{i}}{}_mdx^r-\overline{\Gamma}_{\bar{r}}{}^i{}_m\delta y^r)\delta y^j+2g_{ij}dx^i(-\overline{\Gamma}_r{}^{\bar{j}}{}_mdx^r-\overline{\Gamma}_{\bar{r}}{}^{\bar{j}}{}_m\delta y^r)+\partial_mg_{ij}\delta y^i\delta y^j\\
&\quad+2g_{ij}(-\overline{\Gamma}_r{}^{\bar{i}}{}_mdx^r-\overline{\Gamma}_{\bar{r}}{}^{\bar{i}}{}_m\delta y^r)\delta y^j\\
&=-2g_{ir}\overline{\Gamma}_j{}^{\bar{r}}{}_mdx^idx^j+2(\partial_mg_{ij}-g_{rj}\overline{\Gamma}_i{}^r{}_m-g_{ir}\overline{\Gamma}_{\bar{j}}{}^{\bar{r}}{}_m-g_{rj}\overline{\Gamma}_i{}^{\bar{r}}{}_m)dx^i\delta y^j\\
&\quad+(\partial_mg_{ij}-2g_{rj}\overline{\Gamma}_{\bar{i}}{}^r{}_m-2g_{rj}\overline{\Gamma}_{\bar{i}}{}^{\bar{r}}{}_m)\delta y^i\delta y^j,
\end{aligned}
$$

and

$$
\begin{aligned}
0&=\overline{\nabla}_{X_{\bar{m}}}G\\
&=\overline{\nabla}_{X_{\bar{m}}}(2g_{ij}dx^i\delta y^j+g_{ij}\delta y^i\delta y^j)\\
&=2g_{ij}(\overline{\nabla}_{X_{\bar{m}}}dx^i)\delta y^j+2g_{ij}dx^i(\overline{\nabla}_{X_{\bar{m}}}\delta y^j)+2g_{ij}(\overline{\nabla}_{X_{\bar{m}}}\delta y^i)\delta y^j\\
&=2g_{ij}(-\overline{\Gamma}_r{}^i{}_{\bar{m}}dx^r-\overline{\Gamma}_{\bar{r}}{}^i{}_{\bar{m}}\delta y^r)\delta y^j+2g_{ij}dx^i(-\overline{\Gamma}_r{}^{\bar{j}}{}_{\bar{m}}dx^r-\overline{\Gamma}_{\bar{r}}{}^{\bar{j}}{}_{\bar{m}}\delta y^r)+2g_{ij}(-\overline{\Gamma}_r{}^{\bar{i}}{}_{\bar{m}}dx^r-\overline{\Gamma}_{\bar{r}}{}^{\bar{i}}{}_{\bar{m}}\delta y^r)\delta y^j\\
&=-2g_{ir}\overline{\Gamma}_j{}^{\bar{r}}{}_{\bar{m}}dx^idx^j-2(g_{rj}\overline{\Gamma}_i{}^r{}_{\bar{m}}+g_{ir}\overline{\Gamma}_{\bar{j}}{}^{\bar{r}}{}_{\bar{m}}+g_{rj}\overline{\Gamma}_i{}^{\bar{r}}{}_{\bar{m}})dx^i\delta y^j-2(g_{rj}\overline{\Gamma}_{\bar{i}}{}^r{}_{\bar{m}}+g_{rj}\overline{\Gamma}_{\bar{i}}{}^{\bar{r}}{}_{\bar{m}})\delta y^i\delta y^j.
\end{aligned}
$$

It follows that

(3-3) $g_{ir}\overline{\Gamma}_j{}^{\bar{r}}{}_m+g_{jr}\overline{\Gamma}_i{}^{\bar{r}}{}_m=0$,

(3-4) $\partial_mg_{ij}-g_{ir}\overline{\Gamma}_i{}^r{}_m-g_{ri}\overline{\Gamma}_{\bar{j}}{}^{\bar{r}}{}_m-g_{rj}\overline{\Gamma}_i{}^{\bar{r}}{}_m=0$,

(3-5) $\partial_mg_{ij}-g_{rj}\Gamma_{\bar{i}}{}^r{}_m-g_{ri}\overline{\Gamma}_{\bar{j}}{}^r{}_m-g_{rj}\overline{\Gamma}_{\bar{i}}{}^{\bar{r}}{}_m-g_{ri}\overline{\Gamma}_{\bar{j}}{}^{\bar{r}}{}_m=0$,

(3-6) $g_{ir}\overline{\Gamma}_j{}^{\bar{r}}{}_{\bar{m}}+g_{jr}\overline{\Gamma}_i{}^{\bar{r}}{}_{\bar{m}}=0$,

(3-7) $g_{rj}\overline{\Gamma}_{\bar{i}}{}^r{}_{\bar{m}}+g_{ir}\overline{\Gamma}_{\bar{j}}{}^{\bar{r}}{}_{\bar{m}}+g_{rj}\overline{\Gamma}_i{}^{\bar{r}}{}_{\bar{m}}=0$,

(3-8) $g_{rj}\overline{\Gamma}_{\bar{i}}{}^r{}_{\bar{m}}+g_{ri}\overline{\Gamma}_{\bar{j}}{}^r{}_{\bar{m}}+g_{rj}\overline{\Gamma}_{\bar{i}}{}^{\bar{r}}{}_{\bar{m}}+g_{ri}\overline{\Gamma}_{\bar{j}}{}^{\bar{r}}{}_{\bar{m}}=0$.

From (3-3) and (2) of Lemma 3, we have $g_{ir}\overline{\Gamma}_j{}^{\bar{r}}{}_m=-g_{jr}\overline{\Gamma}_i{}^{\bar{r}}{}_m=-g_{jr}(\overline{\Gamma}_m{}^{\bar{r}}{}_i+y^aK_{ima}{}^r)=g_{mr}\overline{\Gamma}_j{}^{\bar{r}}{}_i-g_{jr}y^aK_{ima}{}^r=g_{mr}(\overline{\Gamma}_i{}^{\bar{r}}{}_j+y^aK_{jia}{}^r)-g_{jr}y^aK_{ima}{}^r=-g_{ir}\overline{\Gamma}_m{}^{\bar{r}}{}_j+y^a(g_{mr}K_{jia}{}^r-g_{jr}K_{ima}{}^r)=-g_{ir}(\overline{\Gamma}_j{}^{\bar{r}}{}_m+y^aK_{mja}{}^r)+y^a(g_{mr}K_{ija}{}^r-g_{jr}K_{ima}{}^r)$, thus we get $2g_{ir}\overline{\Gamma}_j{}^{\bar{r}}{}_m=y^a(g_{mr}K_{jia}{}^r-g_{jr}K_{ima}{}^r-g_{ir}K_{mja}{}^r)$, then we obtain

(3-9) $\overline{\Gamma}_j{}^{\bar{h}}{}_m=y^aK_{amj}{}^h$.

From (3- 5), we have $g_{ir}(\Gamma_j{}^r{}_m-\overline{\Gamma}_{\bar{j}}{}^{\bar{r}}{}_m-\overline{\Gamma}_{\bar{j}}{}^r{}_m)+g_{jr}(\Gamma_i{}^r{}_m-\overline{\Gamma}_{\bar{i}}{}^{\bar{r}}{}_m-\overline{\Gamma}_{\bar{i}}{}^r{}_m)==0$, then by (3) and (4) of Lemma 3, we obtain

(3-10) $g_{ir}(\overline{\Gamma}_m{}^{\bar{r}}{}_{\bar{j}}+\overline{\Gamma}_m{}^r{}_{\bar{j}})+g_{jr}(\overline{\Gamma}_m{}^{\bar{r}}{}_{\bar{i}}+\overline{\Gamma}_m{}^r{}_{\bar{i}})=0$.

Substituting (3-10) into (3-7), we get

(3-11) $\overline{\Gamma}_{\bar{i}}{}^{\bar{h}}{}_{\bar{j}}=0,$

and

(3-12) $\overline{\Gamma}_{i}{}^{h}{}_{\bar{j}}+\overline{\Gamma}_{i}{}^{\bar{h}}{}_{\bar{j}}=0.$

From (3-8), (5) of Lemma 3 and (3-11), we have $g_{jr}\overline{\Gamma}_{\bar{i}}{}^{r}{}_{\bar{m}}=-g_{ir}\overline{\Gamma}_{\bar{j}}{}^{r}{}_{\bar{m}}=-g_{ir}\overline{\Gamma}_{\bar{m}}{}^{r}{}_{\bar{j}}=g_{mr}\overline{\Gamma}_{\bar{i}}{}^{r}{}_{\bar{j}}=g_{mr}\overline{\Gamma}_{\bar{j}}{}^{r}{}_{\bar{i}}=-g_{jr}\overline{\Gamma}_{\bar{m}}{}^{r}{}_{\bar{i}}=-g_{jr}\overline{\Gamma}_{\bar{i}}{}^{r}{}_{\bar{m}}$, thus we obtain

(3-13) $\overline{\Gamma}_{\bar{i}}{}^{h}{}_{\bar{j}}=0.$

From (3-4) and (3-9), we have

(3-14) $g_{ir}(\Gamma_{j}{}^{r}{}_{m}-\overline{\Gamma}_{\bar{j}}{}^{\bar{r}}{}_{m})+g_{jr}(\Gamma_{i}{}^{r}{}_{m}-\overline{\Gamma}_{i}{}^{r}{}_{m})=g_{jr}y^{a}K_{ami}{}^{r}.$

Substituting (4) of Lemma 3 into (3-14), we obtain

(3-15) $g_{ir}\overline{\Gamma}_{m}{}^{\bar{r}}{}_{\bar{j}}=g_{jr}(\Gamma_{i}{}^{r}{}_{m}-\overline{\Gamma}_{i}{}^{r}{}_{m})-g_{jr}y^{a}K_{ami}{}^{r}.$

Substituting (3-15) into (3-6), we get

(3-16) $\overline{\Gamma}_{i}{}^{h}{}_{j}=\Gamma_{i}{}^{h}{}_{j}-\frac{1}{2}y^{a}(K_{aij}{}^{h}+K_{aji}{}^{h}).$

From (3-15) and (3-16), we have

(3-17) $\overline{\Gamma}_{i}{}^{\bar{h}}{}_{\bar{j}}=\frac{1}{2}y^{a}K_{aji}{}^{h}.$

From (3), (4) of Lemma 3, (3-12) and (3-17), we get

(3-18) $\overline{\Gamma}_{\bar{j}}{}^{\bar{h}}{}_{i}=\Gamma_{j}{}^{h}{}_{i}+\frac{1}{2}y^{a}K_{aji}{}^{h},$

and

(3-19) $\overline{\Gamma}_{i}{}^{h}{}_{\bar{j}}=-\frac{1}{2}y^{a}K_{aji}{}^{h}.$

This completes the proof of Proposition. q.e.d.

4. Proof of Theorem

We need the following well known fact to prove Theorem (see [1]).

Lemma 4. *If a complete Riemannian manifold M admits a non-isometric homothetic vector field, then M is locally Euclidean.*

Let X be an infinitesimal fibre-preserving projective transformation of TM. Then X is said to be an infinitesimal fibre-preserving projective transformation, if there exists a 1-form θ of TM such that

$$L_X\overline{\nabla}_Y Z-\overline{\nabla}_Y L_X Z-\overline{\nabla}_{[X,Y]}Z=\theta(Y)Z+\theta(Z)Y,$$

for every vector field Y and Z on TM. Let $(\theta_i,\theta_{\bar{i}})$ be the components of θ with respect to the dual basis $(dx^h,\delta y^h)$. We compute the following three cases:

(4-1) $L_X\overline{\nabla}_{X_{\bar{i}}}X_j-\overline{\nabla}_{X_{\bar{i}}}L_X X_j-\overline{\nabla}_{[X,X_{\bar{i}}]}X_j=\theta(X_{\bar{i}})X_j+\theta(X_j)X_{\bar{i}},$

(4-2) $L_X\overline{\nabla}_{X_{\bar{i}}}X_{\bar{j}}-\overline{\nabla}_{X_{\bar{i}}}L_X X_{\bar{j}}-\overline{\nabla}_{[X,X_{\bar{i}}]}X_{\bar{j}}=\theta(X_{\bar{i}})X_{\bar{j}}+\theta(X_{\bar{j}})X_{\bar{i}},$

(4-3) $L_X\overline{\nabla}_{X_i}X_j-\overline{\nabla}_{X_i}L_X X_j-\overline{\nabla}_{[X,X_i]}X_j=\theta(X_i)X_j+\theta(X_j)X_i.$

Proof of Theorem. Let X be a non-affine infinitesimal fibre-preserving projective transformation of TM with the metric II+III. By means of Lemma 1, Lemma 2, Proposition and (4-1), we have

$$(4\text{-}4)\quad -\frac{1}{2}\{y^r(L_V K_{rij}{}^h - K_{aij}{}^h\nabla_r v^a - K_{raj}{}^h\nabla_i v^a + K_{raj}{}^h X_{\bar{i}}(v^{\bar{a}})) + v^{\bar{a}} K_{aij}{}^h\} = \delta_j^h \theta_{\bar{i}},$$

and

$$(4\text{-}5)\quad \frac{1}{2}y^r y^s v^a K_{amr}{}^h K_{sij}{}^m + \frac{1}{2}y^r(v^{\bar{a}}\Gamma_a{}^h{}_m K_{rij}{}^m + K_{rij}{}^m X_m(v^{\bar{h}}) + L_V K_{rij}{}^h$$
$$+ K_{rij}{}^a\nabla_a v^h - K_{aij}{}^h\nabla_r v^a - K_{raj}{}^h\nabla_i v^a - K_{rij}{}^m X_{\bar{m}}(v^{\bar{h}}) + K_{rmj}{}^h X_{\bar{i}}(v^{\bar{m}}))$$
$$+ \frac{1}{2}v^{\bar{a}} K_{aij}{}^h + v^a K_{aij}{}^h + \Gamma_a{}^h{}_j X_{\bar{i}}(v^{\bar{a}}) + X_{\bar{i}} X_j(v^{\bar{h}}) = \delta_j^h \theta_j,$$

where $L_V K_{rij}{}^h$ denote the components of the Lie derivative of the curvature tensor of M with respect to the induced vector field V and $\nabla_r v^a$ denote the componets of the covariant derivative of V. Contracting h and j in (4-4), we get

$$(4\text{-}6)\quad \theta_{\bar{i}} = 0.$$

By using Lemma 1, Lemma 2, Proposition, (4-2) and (4-6), we have $X_{\bar{i}} X_{\bar{j}}(v^{\bar{h}}) = 0$. It follows that we can put

$$(4\text{-}7)\quad v^{\bar{h}} = y^a A_a^h + B^h,$$

where A_a^h and B^h are certain functions which depend only on variables (x^h), and the coordinate transformation rule implies that A_a^h and B^h are the components of a certain (1, 1) tensor field A of M and a certain contravariant vector field B on M, respectively. Substituting (4-7) into (4-4) and (4-5), we obtain

$$(4\text{-}8)\quad L_V K_{rij}{}^h - K_{aij}{}^h\nabla_r v^a - K_{raj}{}^h\nabla_i v^a + K_{raj}{}^h A_i^a + K_{aij}{}^h A_r^a = 0,$$

$$(4\text{-}9)\quad K_{aij}{}^h B^a = 0,$$

$$(4\text{-}10)\quad \frac{1}{2}y^r y^s K_{rij}{}^m(K_{ams}{}^h v^a + \nabla_m A_s^h) + \frac{1}{2}y^r K_{rij}{}^a(\nabla_a v^h - A_a^h + \nabla_a B^h) + K_{aji}{}^h v^a + \nabla_j A_i^h = \delta_i^h \theta_j,$$

where $\nabla_m A_s^h$ and $\nabla_a B^h$ denote the components of the covariant derivative of A and B, respectively. Contracting y^i into (4-10), we have

$$(4\text{-}11)\quad K_{aji}{}^h v^a + \nabla_j A_i^h = \delta_i^h \theta_j.$$

Substituting (4-11) into (4-10), we get

$$(4\text{-}12)\quad K_{rij}{}^m \theta_m = 0,$$

$$(4\text{-}13)\quad K_{rij}{}^a(\nabla_a v^h - A_a^h + \nabla_a B^h) = 0,$$

By virtue of Lemma 1, Lemma 2, Proposition, (4-3), (4-7), (4-8) and (4-11), we have

$$(4\text{-}14)\quad \nabla_i\nabla_j v^h + K_{aij}{}^h v^a = \delta_i^h \theta_j + \delta_j^h \theta_i,$$

$$(4\text{-}15)\quad \nabla_i \theta_j = 0.$$

Putting $w^h = g^{ab}\theta_b(\nabla_a v^h - A_a^h)$, then by (4-11), (4-12), (4-14) and (4-15), we can show that the vector field W with the components (w^h) is a non-isometric homothetic vector field on M. This completes the proof of Theorem by Lemma 4. q.e.d.

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