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On Infinitesimal Projective Transformations of the Tangent Bundles with the Complete Lift Metric over Riemannian Manifolds

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ABSTRACT. Let M be a non-Euclidean complete n-dimensional Riemannian manifold and TM be its tangent bundle with the complete lift metric. Then every infinitesimal fibre-preserving projective transformation of TM is an affine one.

Introduction

In the present paper everything will be always discussed in the C^∞ category, and Riemannian manifolds will be assumed to be connected and dimension >1.

Let M be a Riemannian manifold, and let ϕ be a transformation of M. Then ϕ is called a projective transformation, if it preserves the geodesics, where each geodesic should be confounded with a subset of M by neglecting its affine parameter. Furthermore ϕ is called an affine transformation, if it preserves the Riemannian connection. We then remark that an affine transformation may be characterized as a projective transformation which preserves the affine parameter together with the geodesics. We may also speak of local projective and affine transformations.

Let V be a vector field on M, and let us consider a local one-parameter group $\{\phi_t\}$ of local transformations of M generated by V. Then V is called an infinitesimal projective (resp.affine) transformation, if each ϕ_t is a local projective (resp.affine) transformation. By a complete infinitesimal projective transformation we mean an infinitesimal projective transformation which generates a global one-parameter group of projective transformations.

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Clearly an infinitesimal affine transformation is an infinitesimal projective transformation. The converse is not true in general. Indeed consider the n-dimensional real projective space $P^n(R)$ with the standard Riemannian metric, which is the standard projectively flat Riemannian manifold, and is a space of positive constant curvature. It is well known that $P^n(R)$ admits a non-affine infinitesimal projective transformation. As a converse problem, the following conjecture is famous.

Conjecture. *Let M be a complete n-dimensional Riemannian manifold admitting a global non-affine infinitesimal projective transformation. Then is M a space of positive constant curvature?*

Let TM be the tangent bundle over M and g be a Riemannian metric of M. Then, by using g, we can define a Riemannian metric or a psuedo-Riemannian metric of TM called the complete lift metric. Let X be a vector field on TM, and let us consider a local one-parameter group $\{\Phi_t\}$ of local transformations of TM generated by X. Then X is called an infinitesimal fibre-preserving transformation, if each Φ_t is a local fibre-preserving transformation of TM.

The purpose of the present paper is to investigate some relations between the above conjecture and the Lie algebra of infinitesimal fibre-preserving projective transformations of TM with the complete lift metric, and we prove the following theorem

Theorem. *Let M be a non-Euclidean complete n-dimensional Riemannian manifold, and let TM be its tangent bundle with the complete lift metric. Then every infinitesimal fibre-preserving projective transformation X of TM is an affine one and it naturally induces an infinitesimal affine transformation V of M. Furthermore the correspondence* $X \rightarrow V$ *gives a homomorphism of the Lie algebra of infinitesimal fibre-preserving projective transformations of TM onto the Lie algebra of infinitesimal affine ones of M.*

§1. Preliminaries

Let $\Gamma_i{}^h{}_j$ be the coefficients of the Riemannian connection of M, then $y^a\Gamma_a{}^h{}_j$ can be regarded as coefficients of a non-linear connection of TM, where (x^h, y^h) the induced coordinates in TM. Using $y^a\Gamma_a{}^h{}_j$, we define a local basis $\{X_h, X_{\bar{h}}\}$ of TM as follows:

$$X_h = \frac{\partial}{\partial x^h} - y^a\Gamma_a{}^m{}_h\frac{\partial}{\partial y^m} \qquad \text{and} \quad X_{\bar{h}} = \frac{\partial}{\partial y^h},$$

then $\{X_h, X_{\bar{h}}\}$ is called the adapted frame of TM, and let $\{dx^h, \delta y^h\}$ be the dual basis of $\{X_h, X_{\bar{h}}\}$.

Lemma 1. *The Lie brackets of the adapted frame of TM satisfy the following:*

(1) $[X_i, X_j] = y^a K_{jia}{}^m X_{\bar{m}}$,

(2) $[X_i, X_{\bar{j}}] = \Gamma_j{}^m{}_i X_{\bar{m}}$,

(3) $[X_{\bar{i}}, X_{\bar{j}}] = 0$,

where $K_{jia}{}^m$ denote the components of the curvature tensor of M.

Proof. By the definition of the adapted frame, we have

$$[X_i, X_j] = [\partial/\partial x^i - y^a \Gamma_a{}^m{}_i X_{\bar{m}}, \partial/\partial x^j - y^b \Gamma_b{}^r{}_j X_{\bar{r}}]$$
$$= y^a (\partial \Gamma_a{}^m{}_i/\partial x^j - \partial \Gamma_a{}^m{}_j/\partial x^i + \Gamma_r{}^m{}_j \Gamma_a{}^r{}_i - \Gamma_r{}^m{}_i \Gamma_a{}^r{}_j) X_{\bar{m}}$$
$$= y^a K_{jia}{}^m X_{\bar{m}}.$$

Thus we obtain (1). We get (2) and (3) in a similar fashion. q.e.d.

Let X be an infinitesimal fibre-preserving transformation of TM and $(v^h, v^{\bar{h}})$ the components of X with respect to the adapted frame $\{X_h, X_{\bar{h}}\}$. The components v^h and $v^{\bar{h}}$ are said to be the horizontal components and the vertical components of X, respectively. It is well known that X is an infinitesimal fibre-preserving transformation if and only if the horizontal components v^h depend only on the variables (x^h). Thus X induces a vector field V with the components v^h in the base space M.

Lemma 2. *Let X be an infinitesimal fibre-preserving transformation of TM with the components $(v^h, v^{\bar{h}})$ and L_X be the Lie derivation with respect to X. Then the Lie derivatives of the adapted frame and the dual basis are given as follows:*

(1) $L_X X_h = -(\partial v^m/\partial x^h) X_m + \{y^a v^b K_{hba}{}^m - v^{\bar{b}} \Gamma_b{}^m{}_h - X_h(v^{\bar{m}})\} X_{\bar{m}}$,

(2) $L_X X_{\bar{h}} = \{v^b \Gamma_b{}^m{}_h - X_{\bar{h}}(v^{\bar{m}})\} X_{\bar{m}}$,

(3) $L_X dx^h = (\partial v^h/\partial x^m) dx^m$,

(4) $L_X \delta y^h = -\{y^a v^b K_{mba}{}^h - v^{\bar{b}} \Gamma_b{}^h{}_m - X_m(v^{\bar{h}})\} dx^m - \{v^b \Gamma_b{}^h{}_m - X_{\bar{m}}(v^{\bar{h}})\} \delta y^m$.

Proof. By Lemma 1 and the definition of Lie derivation, we have

$$L_X X_h = [X, X_h]$$
$$= [v^b X_b + v^{\bar{b}} X_{\bar{b}}, X_h]$$
$$= -(\partial v^m/\partial x^h) X_m + \{y^a v^b K_{hba}{}^m - v^{\bar{b}} \Gamma_b{}^m{}_h - X_h(v^{\bar{m}})\} X_{\bar{m}},$$

thus we obtain (1). We get (2), in a similar fashion. To prove (3), we put $L_X dx^h = A^h{}_m dx^m + B^h{}_m \delta y^m$, then we have $0 = L_X(dx^h(X_m)) = A^h{}_m - \partial v^h/\partial x^m$ and $0 = L_X(dx^h(X_{\bar{m}})) = B^h{}_m$, hence we obtain (3). We get (4), analogously. q.e.d.

§2. The Riemannian connection of *TM* with the complete lift metric

Let $g = g_{ij} dx^i dx^j$ be a Riemannian metric of M, then we can define a Riemannian or a psuedo-Riemannian metric G of TM as follows: $G = 2 g_{ij} dx^i \delta y^j$. We call this metric the complete lift metric. Let $\overline{\nabla}$ be the Riemannian connection of TM with the complete lift metric and $\overline{\Gamma}_B{}^A{}_C$ the coefficients of $\overline{\nabla}$, that is,

$$(2.1)\quad \begin{aligned} &\overline{\nabla}_{X_i}X_j=\overline{\Gamma}_j{}^m{}_iX_m+\overline{\Gamma}_j{}^{\bar m}{}_iX_{\bar m}, &&\overline{\nabla}_{X_i}X_{\bar j}=\overline{\Gamma}_{\bar j}{}^m{}_iX_m+\overline{\Gamma}_{\bar j}{}^{\bar m}{}_iX_{\bar m},\\ &\overline{\nabla}_{X_{\bar i}}X_j=\overline{\Gamma}_j{}^m{}_{\bar i}X_m+\overline{\Gamma}_j{}^{\bar m}{}_{\bar i}X_m, &&\overline{\nabla}_{X_{\bar i}}X_{\bar j}=\overline{\Gamma}_{\bar j}{}^m{}_{\bar i}X_m+\overline{\Gamma}_{\bar j}{}^{\bar m}{}_{\bar i}X_{\bar m}.\end{aligned}$$

Lemma 3. *We have the following equations.*

(1) $\overline{\nabla}_{X_i}dx^h=-\overline{\Gamma}_m{}^h{}_idx^m-\overline{\Gamma}_{\bar m}{}^h{}_i\delta y^m$,

(2) $\overline{\nabla}_{X_i}\delta y^h=-\overline{\Gamma}_m{}^{\bar h}{}_idx^m-\overline{\Gamma}_{\bar m}{}^{\bar h}{}_i\delta y^m$,

(3) $\overline{\nabla}_{X_{\bar i}}dx^h=-\overline{\Gamma}_m{}^h{}_{\bar i}dx^m-\overline{\Gamma}_{\bar m}{}^h{}_{\bar i}\delta y^m$,

(4) $\overline{\nabla}_{X_{\bar i}}\delta y^h=-\overline{\Gamma}_m{}^{\bar h}{}_{\bar i}dx^m-\overline{\Gamma}_{\bar m}{}^{\bar h}{}_{\bar i}\delta y^m$.

Proof. We put $\overline{\nabla}_{X_i}dx^h=A_m{}^h{}_idx^m+B_m{}^h{}_i\delta y^m$, then $0=\overline{\nabla}_{X_i}(dx^h(X_j))=A_j{}^h{}_i+\overline{\Gamma}_j{}^h{}_i$ and $0=\overline{\nabla}_{X_i}(dx^h(X_{\bar j}))=B_j{}^h{}_i+\overline{\Gamma}_{\bar j}{}^h{}_i$, thus we get (1). Similarly, we obtain (2), (3) and (4). q.e.d.

Since the torsion tensor $T(X,Y)$ of $\overline{\nabla}$ defined by $T(X,Y)=\overline{\nabla}_XY-\overline{\nabla}_YX-[X,Y]$ vanishes, we have the following relations by means of Lemma 1 and (2-1).

$$(2.2)\quad \begin{aligned} &\overline{\Gamma}_j{}^h{}_i=\overline{\Gamma}_i{}^h{}_j, &&\overline{\Gamma}_j{}^{\bar h}{}_i=\overline{\Gamma}_i{}^{\bar h}{}_j+y^aK_{jia}{}^h,\\ &\overline{\Gamma}_{\bar j}{}^h{}_i=\overline{\Gamma}_i{}^h{}_{\bar j}, &&\overline{\Gamma}_{\bar j}{}^{\bar h}{}_i=\overline{\Gamma}_i{}^{\bar h}{}_{\bar j}+\Gamma_j{}^h{}_i,\\ &\overline{\Gamma}_{\bar j}{}^h{}_{\bar i}=\overline{\Gamma}_{\bar i}{}^h{}_{\bar j}, &&\overline{\Gamma}_{\bar j}{}^{\bar h}{}_{\bar i}=\overline{\Gamma}_{\bar i}{}^{\bar h}{}_{\bar j}.\end{aligned}$$

Furthermore we have the following lemma.

Lemma 4. *The connection coefficients $\overline{\Gamma}_B{}^A{}_C$ of $\overline{\nabla}$ satisfy the following relations.*

(1) $\overline{\Gamma}_j{}^h{}_i=\Gamma_i{}^h{}_j$, (2) $\overline{\Gamma}_j{}^{\bar h}{}_i=y^aK_{aij}{}^h$,

(3) $\overline{\Gamma}_{\bar j}{}^h{}_i=0$, (4) $\overline{\Gamma}_j{}^h{}_{\bar i}=0$,

(5) $\overline{\Gamma}_{\bar j}{}^{\bar h}{}_i=\Gamma_i{}^h{}_j$, (6) $\overline{\Gamma}_j{}^{\bar h}{}_{\bar i}=0$,

(7) $\overline{\Gamma}_{\bar j}{}^h{}_{\bar i}=0$, (8) $\overline{\Gamma}_{\bar j}{}^{\bar h}{}_{\bar i}=0$.

Proof. By means of lemma 3 and the connection $\overline{\nabla}$ is metrical, that is $\overline{\nabla}G=0$, we have

$$\begin{aligned} 0&=\overline{\nabla}_{X_m}G\\ &=\overline{\nabla}_{X_m}(2g_{ij}dx^i\delta y^j)\\ &=-(g_{ir}\overline{\Gamma}_j{}^{\bar r}{}_m+g_{jr}\overline{\Gamma}_i{}^{\bar r}{}_m)dx^idx^j+2\{g_{ir}(\Gamma_j{}^r{}_m-\overline{\Gamma}_{\bar j}{}^{\bar r}{}_m)+g_{jr}(\Gamma_i{}^r{}_m-\overline{\Gamma}_i{}^r{}_m)\}dx^i\delta y^j\\ &\quad-(g_{ir}\overline{\Gamma}_{\bar j}{}^r{}_m+g_{jr}\overline{\Gamma}_{\bar i}{}^r{}_m)\delta y^i\delta y^j\end{aligned}$$

and

$$\begin{aligned} 0&=\overline{\nabla}_{X_{\bar m}}G\\ &=\overline{\nabla}_{X_{\bar m}}(2g_{ij}dx^i\delta y^j)\\ &=-(g_{ir}\overline{\Gamma}_j{}^{\bar r}{}_{\bar m}+g_{jr}\overline{\Gamma}_i{}^{\bar r}{}_{\bar m})dx^idx^j-2(g_{ir}\overline{\Gamma}_{\bar j}{}^{\bar r}{}_{\bar m}+g_{jr}\overline{\Gamma}_i{}^r{}_{\bar m})dx^i\delta y^j-(g_{ir}\overline{\Gamma}_{\bar j}{}^r{}_{\bar m}+g_{jr}\overline{\Gamma}_{\bar i}{}^r{}_{\bar m})\delta y^i\delta y^j,\end{aligned}$$

it follows that

$$(2\text{-}3)\quad g_{ir}\overline{\Gamma}_j{}^{\bar r}{}_m+g_{jr}\overline{\Gamma}_i{}^{\bar r}{}_m=0,$$

(2-4) $g_{ir}(\Gamma_{j}{}^{r}{}_{m}-\overline{\Gamma}_{\bar{j}}{}^{\bar{r}}{}_{m})+g_{jr}(\Gamma_{i}{}^{r}{}_{m}-\overline{\Gamma}_{i}{}^{r}{}_{m})=0,$

(2-5) $g_{ir}\overline{\Gamma}_{\bar{j}}{}^{r}{}_{m}+g_{jr}\overline{\Gamma}_{\bar{i}}{}^{r}{}_{m}=0,$

(2-6) $g_{ir}\overline{\Gamma}_{j}{}^{\bar{r}}{}_{\bar{m}}+g_{jr}\overline{\Gamma}_{i}{}^{\bar{r}}{}_{\bar{m}}=0,$

(2-7) $g_{ir}\overline{\Gamma}_{\bar{j}}{}^{\bar{r}}{}_{\bar{m}}+g_{jr}\overline{\Gamma}_{i}{}^{r}{}_{\bar{m}}=0,$

(2-8) $g_{ir}\overline{\Gamma}_{\bar{j}}{}^{r}{}_{\bar{m}}+g_{jr}\overline{\Gamma}_{\bar{i}}{}^{r}{}_{\bar{m}}=0.$

From (2-2) and (2-3), we have

$$g_{ir}\overline{\Gamma}_{j}{}^{\bar{r}}{}_{m}=-g_{jr}\overline{\Gamma}_{i}{}^{\bar{r}}{}_{m}=-g_{jr}(\overline{\Gamma}_{m}{}^{\bar{r}}{}_{i}+y^{a}K_{ima}{}^{r})=g_{mr}\overline{\Gamma}_{j}{}^{\bar{r}}{}_{i}-y^{a}K_{imaj}$$
$$=g_{mr}(\overline{\Gamma}_{i}{}^{\bar{r}}{}_{j}+y^{a}K_{jia}{}^{r})-y^{a}K_{imaj}=-g_{ir}\overline{\Gamma}_{m}{}^{\bar{r}}{}_{j}+y^{a}K_{jiam}-y^{a}K_{imaj}$$
$$=-g_{ir}(\overline{\Gamma}_{j}{}^{\bar{r}}{}_{m}+y^{a}K_{mja}{}^{r})+y^{a}(K_{amji}+K_{ajmi})=-g_{ir}\overline{\Gamma}_{j}{}^{\bar{r}}{}_{m}+y^{a}K_{aijm}+y^{a}(K_{amji}+K_{ajmi})$$
$=-g_{ir}\overline{\Gamma}_{j}{}^{\bar{r}}{}_{m}+2y^{a}K_{amji}$, thus we get (2).

From (2-2) and (2-8), we have

$g_{ir}\overline{\Gamma}_{\bar{j}}{}^{r}{}_{\bar{m}}=-g_{jr}\overline{\Gamma}_{\bar{i}}{}^{r}{}_{\bar{m}}=-g_{jr}\overline{\Gamma}_{\bar{m}}{}^{r}{}_{\bar{i}}=g_{mr}\overline{\Gamma}_{\bar{j}}{}^{r}{}_{\bar{i}}=g_{mr}\overline{\Gamma}_{\bar{i}}{}^{r}{}_{\bar{j}}=-g_{ir}\overline{\Gamma}_{\bar{m}}{}^{r}{}_{\bar{j}}=-g_{ir}\overline{\Gamma}_{\bar{j}}{}^{r}{}_{\bar{m}}$, thus we get (7). From (2-2) and (2-4), we have $g_{ir}(\Gamma_{j}{}^{r}{}_{m}-\overline{\Gamma}_{j}{}^{r}{}_{m})=-g_{jr}(\Gamma_{i}{}^{r}{}_{m}-\overline{\Gamma}_{\bar{i}}{}^{\bar{r}}{}_{m})=g_{jr}\overline{\Gamma}_{m}{}^{\bar{r}}{}_{\bar{i}}$, thus from (2-6), we get $g_{ir}(\Gamma_{j}{}^{r}{}_{m}-\overline{\Gamma}_{j}{}^{r}{}_{m})+g_{ir}(\Gamma_{m}{}^{r}{}_{j}-\overline{\Gamma}_{m}{}^{r}{}_{j})=g_{jr}\overline{\Gamma}_{m}{}^{\bar{r}}{}_{\bar{i}}+g_{mr}\overline{\Gamma}_{j}{}^{\bar{r}}{}_{\bar{i}}=0.$ This shows(1), (5) and (6). From (2-2) and (2-5) and (2-7), we have

$$0=g_{ir}\overline{\Gamma}_{\bar{j}}{}^{r}{}_{m}=+g_{jr}\overline{\Gamma}_{\bar{i}}{}^{r}{}_{m}=g_{ir}\overline{\Gamma}_{m}{}^{r}{}_{\bar{j}}+g_{jr}\overline{\Gamma}_{m}{}^{r}{}_{\bar{i}}=-(g_{mr}\overline{\Gamma}_{\bar{i}}{}^{\bar{r}}{}_{\bar{j}}+g_{mr}\overline{\Gamma}_{\bar{j}}{}^{\bar{r}}{}_{\bar{i}})=-2g_{mr}\overline{\Gamma}_{\bar{i}}{}^{\bar{r}}{}_{\bar{j}}.$$

This shows (8), (3) and (4). q.e.d.

From lemma 4 and (2-1) we have the following equations.

$$\overline{\nabla}_{X_i}X_j=\Gamma_{j}{}^{h}{}_{i}X_h+y^{a}K_{aij}{}^{h}X_{\bar{h}},$$

(2-9) $\overline{\nabla}_{X_i}X_{\bar{j}}=\Gamma_{j}{}^{h}{}_{i}X_{\bar{h}},$

$$\overline{\nabla}_{X_{\bar{i}}}X_j=0,$$
$$\overline{\nabla}_{X_{\bar{i}}}X_{\bar{j}}=0.$$

§3. Projective transformations of *TM* with the complete lift metric

Let X be an infinitesimal fibre-preserving projective transformation of TM. It is well known that X is an infinitesimal projective transformation if and only if there exists a 1-form θ of TM such that

$$L_X\overline{\nabla}_YZ-\overline{\nabla}_YL_XZ-\overline{\nabla}_{[X,Y]}Z=\theta(Y)Z+\theta(Z)Y$$

for every vector field Y and Z on TM. Let $(v^h, v^{\bar{h}})$ and $(\theta_i, \theta_{\bar{i}})$ be the components of X and θ, respectively. Then $X=v^hX_h+v^{\bar{h}}X_{\bar{h}}$ and $\theta=\theta_i dx^i+\theta_{\bar{i}}\delta y^i$. We compute the following three cases:

(3-1) $L_X\overline{\nabla}_{X_{\bar{i}}}X_j-\overline{\nabla}_{X_{\bar{i}}}L_XX_j-\overline{\nabla}_{[X,X_{\bar{i}}]}X_j=\theta(X_{\bar{i}})X_j+\theta(X_j)X_{\bar{i}},$

(3-2) $L_X\overline{\nabla}_{X_{\bar{i}}}X_{\bar{j}}-\overline{\nabla}_{X_{\bar{i}}}L_XX_{\bar{j}}-\overline{\nabla}_{[X,X_{\bar{i}}]}X_{\bar{j}}=\theta(X_{\bar{i}})X_{\bar{j}}+\theta(X_{\bar{j}})X_{\bar{i}},$

(3-3) $L_X\overline{\nabla}_{X_i}X_j-\overline{\nabla}_{X_i}L_XX_j-\overline{\nabla}_{[X,X_i]}X_j=\theta(X_i)X_j+\theta(X_j)X_i.$

By means of Lemma 1, Lemma 2, (2-9) and (3-1), we have

Left hand side of (3-1) $= -\{v^a K_{jai}{}^h - X_{\bar{i}}(v^{\bar{a}})\Gamma_a{}^h{}_j - X_{\bar{i}} X_j (v^{\bar{h}})\} X_{\bar{h}}$,

Right hand side of (3-1) $= \delta_j^h \theta_{\bar{i}} X_h + \delta_i^h \theta_j X_{\bar{h}}$.

Thus we obtain

(3-4) $\theta_{\bar{i}} = 0$,

and

(3-5) $v^a K_{jai}{}^h - X_{\bar{i}}(v^{\bar{a}})\Gamma_a{}^h{}_j - X_{\bar{i}} X_j (v^{\bar{h}}) = \delta_i^h \theta_i$.

By means of Lemma 1, Lemma 2, (2-9), (3-4) and (3-2), we have

Left hand side of (3-2) $= X_{\bar{i}} X_{\bar{j}} (v^{\bar{h}}) X_{\bar{h}}$,

Right hand side of (3-2) $= (\delta_j^h \theta_{\bar{i}} + \delta_i^h \theta_{\bar{j}}) X_{\bar{h}} = 0$.

Thus we get $X_{\bar{i}} X_{\bar{j}} (v^{\bar{h}}) = 0$, hence we can put

(3-6) $v^{\bar{h}} = y^r A_r^h + B^h$,

where A_r^h and B^h are certain functions which depend only on the variables (x^h) and the coordinate transformation rule implies that A_r^h and B^h are the components of a (1, 1) tensor field A and a contravariant vector field B on M, respectively. Substituting (3-6) into (3-5), we obtain

(3-7) $K_{aji}{}^h v^a + \nabla_j A_i^h = \delta_i^h \theta_j$,

where $\nabla_j A_i^h$ the components of the covariant derivative of A.

By means of Lemma 1, Lemma 2, (2-9), (3-6), (3-7) and (3-3), we have

Left hand side of (3-3)

$= \{\nabla_i \nabla_j v^h + K_{aij}{}^h v^a\} X_h + \{\nabla_i \nabla_j B^h + K_{aij}{}^h B^a$

$+ y^r (\nabla_i \nabla_j A_r^h + A_r^a K_{aij}{}^h - A_a^h K_{rij}{}^a + v^a \nabla_a K_{rij}{}^h - v^a \nabla_i K_{jar}{}^h + \nabla_j v^a K_{ria}{}^h + \nabla_i v^a K_{rja}{}^h\} X_{\bar{h}}$,

Right hand side of (3-3) $= (\delta_j^h \theta_i + \delta_i^h \theta_j) X_h$,

where $\nabla_j v^h$, $\nabla_j B^h$ and $\nabla_a K_{rij}{}^h$ denote the components of covariant derivative of V, B and the curvature tensor of M, respectively.

Hence we have

(3-8) $\nabla_i \nabla_j v^h + K_{aij}{}^h v^a = \delta_j^h \theta_i + \delta_i^h \theta_j$,

(3-9) $\nabla_i \nabla_j B^h + K_{aij}{}^h B^a = 0$,

(3-10) $\nabla_i \nabla_j A_r^h + A_r^a K_{aij}{}^h - A_a^h K_{rij}{}^a + v^a \nabla_a K_{rij}{}^h - v^a \nabla_i K_{jar}{}^h + \nabla_j v^a K_{ria}{}^h + \nabla_i v^a K_{rja}{}^h = 0$.

Proof of Theorem. To prove Theorem, we need the following well known fact.

Lemma 5. ([3]). *If a complete Riemannian manifold M admits a non-isometric homothetic vector field, then M is locally Euclidean.*

The equatin (3-8) shows that the induced vector field $V = v^h \partial / \partial x^h$ is an infinitesimal projective transformation. Hence we obtain $L_V K_{ij} = -(n-1)\nabla_i \theta_j$. Contracting h and

r in (3-10) and using (3-7), we get $\nabla_i \theta_j = 0$ which show $\theta_a \theta^a = $constant and $K_{ijk}{}^a \theta_a = 0$, where $\theta^i = g^{ia} \theta_a$. Putting $w^h = \theta^a \nabla_a v^h - \theta^a A^h_a$, then by (3-7) and (3-8), we have $\nabla_j w_i + \nabla_i w_j = 2\theta_a \theta^a g_{ji}$ where $w_i = g_{ia} w^a$. This shows the vector field W with the components (w^h) is a homothetic vector field, thus by Lemma 5, we have $\theta_i = 0$. Hence X is an infinitesimal fibre-preserving affine transformation of TM, and X naturally induces an infinitesimal affine transformation $V = v^h \frac{\partial}{\partial x^h}$ of M. Conversely, let $V = v^h \frac{\partial}{\partial x^h}$ be an infinitesimal affine transformation of M. We put $X = v^h X_h + y^a \nabla_a v^h X_{\bar{h}}$, then X is an infinitesimal fibre-preserving affine transformation of TM by means of (3-7), (3-8) and (3-10). Therefore the correspondence $X \to V$ gives a homomorphism of the Lie algebra of infinitesimal fibre-preserving projective transformations of TM onto the Lie algebra of infinitesimal affine ones of M. This completes the proof of Theorem.

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