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On infinitesimal conformal transformations of the tangent bundles with the metric I+II over Riemannian manifolds

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Introduction. Let M be an n-dimensional Riemannian manifold with a metric g and let V be a vector field on M. Let us consider the local one-parameter group $\{\phi_t\}$ of local transformations of M generated by V. Then V is called an infinitesimal projective transformation, if each ϕ_t is a local projective transformation of M. It is well known that V is an infinitesimal projective transformation if and only if there exists a covariant vector field ξ on M with the components ξ_i such that $\mathcal{L}_V \Gamma_j{}^h{}_i = \delta^h_j \xi_i + \delta^h_i \xi_j$, where $\mathcal{L}_V$ denotes the Lie derivation with respect to V and $\Gamma_j{}^h{}_i$ the coefficients of the Riemannian connection of M.

Let $T(M)$ be the tangent bundle over M, and let Φ be a transformation of $T(M)$. Then Φ is called a fibre-preserving transformation, if it preserves the fibres. Let X be a vector field on $T(M)$, and let us consider the local one-parameter group $\{\Phi_t\}$ of local transformations of $T(M)$ generated by X. Then X is called an infinitesimal fibre-preserving transformation, if each Φ_t is a local fibre-preserving transformation of $T(M)$. Clearly an infinitesimal fibre-preserving transformation on $T(M)$ induces an infinitesimal transformation in the base space M. An infinitesimal fibre-preserving transformation X on $T(M)$ is called an infinitesimal fibre-preserving conformal transformation, if each Φ_t is a local fibre-preserving conformal transformation of $T(M)$. Let G be a Riemannian or a pseudo-Riemannian metric of $T(M)$. It is well known that X is an infinitesimal conformal transformation of $T(M)$ if and only if there exists a scalar function Ω on $T(M)$ such that $\mathcal{L}_X G = 2\Omega G$, where $\mathcal{L}_X$ denotes the Lie derivation with respect to the vector field X.

In the previous papers [1], [2], we proved the following theorems.

Theorem. *Let M be an n-dimensional Riemannian manifold, and let T(M) be its tangent bundle with the metric* II. *Then every infinitesiml fibre-preserving conformal transformation X on T(M) naturally induces an infinitesimal projective transformation V on M. Furthermore the correspondence $X \to V$ gives a homomorphism of the Lie algebra of infinitesimal fibre-preserving conformal transformations on T(M) onto the Lie algebra of infinitesimal projective transformations on M, and the kernel of this homomorphism is naturally isomorphic onto the Lie algebra of infinitesimal isometries of M.*

Theorem. *Let M be an n-dimnsional Riemannian manifold, and let T(M) be its tangent bundle with the metric* I + III. *Then every infinitesimal fibre-preserving conformal transformation X is a homothetic one and it induces an infinitesimal homothetic transformation V on M. Furthermore the correspondence $X \to V$ gives a homomorphism of the Lie algebra of infinitesimal fibre-preserving conformal transformations on T(M) onto the Lie algebra of infinitesimal homothetic transformations on M, and the kernel of this homomorphism is naturally isomorphic onto the the Lie algebra of infinitesimal isometries of M.*

The purpose of the present paper is to prove the followiing theorem.

Theorem. *Let M be an n-dimensional Riemannian manifold, and let T(M) be its tangent bundle with the metric* I + II. *Then every infinitesiml fibre-preserving conformal transformation X on T(M) naturally induces an infinitesimal projective transformation V on M. Furthermore the correspondence $X \to V$ gives a homomorphism of the Lie algebra of infinitesimal fibre-preserving conformal transformations on T(M) into the Lie algebra of infinitesimal projective transformations on M.*

§ 1. Preliminaries.

Let $\Gamma_j{}^h{}_i$ be the coefficients of the Riemannian connection of M, then $y^a\Gamma_a{}^h{}_i$ can be regarded as coefficients of the non-linear connection of $T(M)$, where (x^h, y^h) the induced coordinates in $T(M)$. We define

$$X_h = \frac{\partial}{\partial x^h} - y^a\Gamma_a{}^m{}_h\frac{\partial}{\partial y^m} \quad \text{and} \quad X_{\bar{h}} = \frac{\partial}{\partial y^h}$$

then $\{X_h, X_{\bar{h}}\}$ are called the adapted frame of $T(M)$, and let $\{dx^h, \delta y^h\}$ be the dual basis of $\{X_h, X_{\bar{h}}\}$.

We can easily prove the following lemma.

Lemma 1. *The Lie brackets satisfy the following:*

$$[X_i, X_j] = y^r K_{jir}{}^m X_{\bar{m}},$$

$$[X_i, X_{\bar{j}}] = \Gamma_j{}^m{}_i X_{\bar{m}},$$

$$[X_{\bar{i}}, X_{\bar{j}}] = 0,$$

where $K_{jir}{}^m$ denote the components of the curvature tensor of M.

Let X be an infinitesimal fibre-preserving transformation on $T(M)$ and $(v^h, v^{\bar{h}})$ the components of X with respect to the adapted frame $\{X_h, X_{\bar{h}}\}$. Then X is fibre-preserving if and only if v^h depend only on the variables (x^h). Clearly X induces an infinitesimal transformation V with the components v^h in the base space M. Let $\mathcal{L}_X$ be the Lie derivation with respect to X, then we have the following lemma.

Lemma 2. (*See* [1]). *The Lie derivatives of the adapted frame and the dual basis are given as follows:*

(1) $\mathcal{L}_X X_h = -\partial_h v^a X_a + \{y^b v^c K_{hcb}{}^a - v^{\bar{b}}\Gamma_b{}^a{}_h - X_h(v^{\bar{a}})\} X_{\bar{a}},$

(2) $\mathcal{L}_X X_{\bar{h}} = \{v^b\Gamma_b{}^a{}_h - X_{\bar{h}}(v^{\bar{a}})\} X_{\bar{a}},$

(3) $\mathcal{L}_X dx^h = \partial_m v^h dx^m,$

(4) $\mathcal{L}_X \delta y^h = -\{y^b v^c K_{mcb}{}^h - v^{\bar{b}}\Gamma_b{}^h{}_m - X_m(v^{\bar{h}})\} dx^m - \{v^b\Gamma_b{}^h{}_m - X_{\bar{m}}(v^{\bar{h}})\} \delta y^m.$

Let g be a Riemannian metric of M with components g_{ji}, then we see that

$$\text{I} : G_{\text{I}} = g_{ji} dx^j dx^i,$$

$$\text{II} : G_{\text{II}} = 2g_{ji}dx^j\delta y^i,$$

$$\text{III} : G_{\text{III}} = g_{ji}\delta y^j\delta y^i,$$

are all quadratic differential forms defiend globally in $T(M)$ and that

$$\text{II} : 2g_{ji}dx^j\delta y^i,$$

$$\text{I} + \text{II} : g_{ji}dx^jdx^i + 2g_{ji}dx^j\delta y^i,$$

$$\text{I} + \text{III} : g_{ji}dx^jdx^i + g_{ji}\delta y^j\delta y^i,$$

$$\text{II} + \text{III} : 2g_{ji}dx^j\delta y^i + g_{ji}\delta y^j\delta y^i,$$

are all non-singular and consequently can be regarded as Riemannian or pseudo-Riemannian metrics in $T(M)$.

Lemma 3. (*See* [1]). *The Lie derivatives* $\mathcal{L}_X G_{\text{I}}$, $\mathcal{L}_X G_{\text{II}}$ *and* $\mathcal{L}_X G_{\text{III}}$ *are given as follows:*

(1) $\mathcal{L}_X G_{\text{I}} = (\mathcal{L}_V g_{ji})dx^jdx^i,$

(2) $\frac{1}{2}\mathcal{L}_X G_{\text{II}} = -g_{jm}\{y^bv^cK_{icb}{}^m - v^{\bar{b}}\Gamma_b{}^m{}_i - X_i(v^{\bar{m}})\}\,dx^jdx^i + \{\mathcal{L}_V g_{ji} - g_{jm}\nabla_i v^m + g_{jm}X_{\bar{i}}(v^{\bar{m}})\}\,dx^j\delta y^i,$

(3) $\mathcal{L}_X G_{\text{III}} = -2g_{mi}\{y^bv^cK_{jcb}{}^m - v^{\bar{b}}\Gamma_b{}^m{}_j - X_j(v^{\bar{m}})\}\,dx^j\delta y^i + \{\mathcal{L}_V g_{ji} - 2g_{mj}\nabla_i v^m + 2g_{mj}X_{\bar{i}}(v^{\bar{m}})\}\,\delta y^j\delta y^i,$

where $\mathcal{L}_V g_{ji}$ *denote the components of the Lie derivative* $\mathcal{L}_V g$ *and* $\nabla_i v^m$ *the components of the covariant derivative of* V.

§ 2. Infinitesimal conformal transformations of the tangent bundles with the metric I + II.

Let $T(M)$ be the tangent bundle over M with the metric I + II, and let X be an infinitesimal fibre-preserving conformal transformation on $T(M)$, that is, there exists a scalar function Ω on $T(M)$ such that $\mathcal{L}_X G_{\text{I}+\text{II}} = 2\Omega G_{\text{I}+\text{II}}$.

Then from *Lemma 3*, we have

$$(2.1)\quad \mathcal{L}_V g_{ji} - 2\Omega g_{ji} = g_{jm}(\nabla_i v^m - X_{\bar{i}}(v^{\bar{m}})),$$

and

$$(2.2)\quad \mathcal{L}_V g_{ji} - 2\Omega g_{ji} = g_{jm}(y^bv^cK_{icb}{}^m - v^{\bar{b}}\Gamma_b{}^m{}_i - X_i(v^{\bar{m}})) + g_{mi}(y^bv^cK_{jcb}{}^m - v^{\bar{b}}\Gamma_b{}^m{}_j - X_j(v^{\bar{m}})).$$

Proposition 1. *The scalar function Ω on $T(M)$ depends only on the variables (x^h) with respect to the induced coordinates (x^h, y^h).*

Proof. Applying $X_{\bar{k}}$ to the both sides of the equation (2. 1), we have

$$2X_{\bar{k}}(\Omega)g_{ji} = g_{jm}X_{\bar{k}}X_{\bar{i}}(\bar{v^m}),$$

from which we get

$$X_{\bar{k}}(\Omega)g_{ji} = X_{\bar{i}}(\Omega)g_{jk},$$

it follows that

$$(n-1)X_{\bar{k}}(\Omega) = 0.$$

This means the scalar function Ω on $T(M)$ depends only on the variables (x^h) with respect to the induced coordinates (x^h, y^h). *Q. E. D.*

Thus we can regard Ω is a function on M, in the following we write ρ instead of Ω.

From (2.1) and *Proposition 1*, $X_{\bar{i}}(\bar{v^m})$ depend only on the variables (x^h), thus we can put

(2. 3) $\qquad \bar{v^h} = y^a A^h{}_a + B^h,$

where $A^h{}_a$ and B^h are certain functions which depend only on the variables (x^h). Furthermore we can show that $A^h{}_a$ and B^h are the components of a (1, 1) tensor field and a contravariant vector field on M, respectively.

Substituting (2. 3) into (2. 1) and (2. 2), we have

(2. 4) $\qquad \mathcal{L}_V g_{ji} - 2\rho g_{ji} - g_{jm}\nabla_i v^m + g_{jm}A^m{}_i = 0,$

(2. 5) $\qquad \mathcal{L}_V g_{ji} - 2\rho g_{ji} + g_{jm}\nabla_i B^m + g_{im}\nabla_j B^m = 0,$

and

(2. 6) $\qquad v^a(K_{aikj} + K_{ajki}) + g_{jm}\nabla_i A^m{}_k + g_{mi}\nabla_j A^m{}_k = 0,$

where $\nabla_i B^m$ and $\nabla_i A^m{}_k$ denote the components of the covariant derivative of the vector field $B = (B^h)$ and the (1, 1) tensor field $A = (A^h{}_i)$, respectively.

Propositon 2. *The vector field V with the components (v^h) is an infinitesimal projective transformation on M.*

Proof. From (2. 4), we obtain

$$
\begin{aligned}
g_{jm}\nabla_k A^m{}_i &= \nabla_k(2\rho g_{ji}+g_{jm}\nabla_i v^m-\mathcal{L}_V g_{ji}) \\
&= 2\rho_k g_{ji}+g_{jm}\nabla_k\nabla_i v^m-\nabla_k\mathcal{L}_V g_{ji} \\
&= 2\rho_k g_{ji}+g_{jm}(\mathcal{L}_V\Gamma_k{}^m{}_i-K_{aki}{}^m v^a)-(\mathcal{L}_V\nabla_k g_{ji}+\mathcal{L}_V\Gamma_k{}^a{}_j g_{ai}+\mathcal{L}_V\Gamma_k{}^a{}_i g_{ja}) \\
&= 2\rho_k g_{ji}-K_{akij}v^a-\mathcal{L}_V\Gamma_k{}^a{}_j g_{ai}.
\end{aligned}
$$

Substituting the above equation into (2. 6), we have

$$\mathcal{L}_V\Gamma_j{}^h{}_i = \delta^h{}_j\rho_i+\delta^h{}_i\rho_j.$$

Hence V is an infinitesimal projective transformation on M. *Q. E. D.*

From *Proposition 2*, the correspondence $X\rightarrow V$ gives a homomorphism of the Lie algebra of infinitesimal fibre-preserving conformal transformations on $T(M)$ into the Lie algebra of infinitesimal projective transformations on M. This shows the proof of the thorem.

If we put $W = V+B$, then W is an infitesimal conformal transformation on M by (2. 5). Therefore if $T(M)$ admits an infinitesimal fibre-preserving conformal transformation, then the base space M admits an infinitersimal projective transformation and an infinitesimal conformal transformation. Conversely, we suppose M admits an infinitesimal conformal transformation W and an infinitesimal projective transformation V such that $\mathcal{L}_V g_{ji} = 2\rho g_{ji}$ and $\mathcal{L}_V\Gamma_j{}^h{}_i = \delta^h{}_j\rho_i+\delta^h{}_i\rho_j$, respectively. Then we can prove the vector field X on $T(M)$ defined by

$$X = v^h X_h+(y^a A^h{}_a+B^h)X_{\bar{h}}$$

is an infinitesimal fibre-preserving conformal transformation on $T(M)$, where v^h and B^h are the components of V and $W-V$, and $A^h{}_a$ are defined by

$$A^h{}_a = g^{hm}(2\rho g_{ma}+\nabla_a v_m-\mathcal{L}_V g_{ma}).$$

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