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On infinitesimal conformal transformations of the tangent bundle with the metric I+III over a Riemannian manifold

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Introduction. In the present paper everything will be always discussed in the C^{∞} category, and Riemannian manifolds will be assumed to be connected and dimension>1.

Let M be a Riemannian manifold with a metric g and let V be a vector field on M. Let us consider the local one-parameter group $\{\phi_t\}$ of local transformations of M generated by V. Then V is called an infinitesimal conformal transformation, if each ϕ_t is a local conformal transformation of M. It is well known that V is an infinitesimal conformal transformation if and only if there exists a scalar function ρ on M such that $\mathcal{L}_V g=2\rho g$, where $\mathcal{L}_V$ denotes the Lie derivation with respect to V, especially V is called an infinitesimal homothetic one when ρ is constant.

Let $T(M)$ be the tangent bundle over M, and let Φ be a transformation of $T(M)$. Then Φ is called a fibre-preserving transformation, if it preserves the fibres. Let X be a vector field on $T(M)$, and let us consider the local one-parameter group $\{\Phi_t\}$ of local transformations of $T(M)$ generated by X. Then X is called an infinitesimal fibre-preserving transformation, if each Φ_t is a local fibre-preserving transformtion of $T(M)$. Clearly an infinitesimal fibre-preserving transformation on $T(M)$ induces an infinitesimal transformation in the base space M. Let G be a Riemannian or a pseudo-Riemannian metric of $T(M)$. An infinitesimal fibre-preserving transformation X on $T(M)$ is said to be an infinitesimal fibre-preserving conformal transformation, if there exists a scalar function Ω on $T(M)$ such that $\mathcal{L}_X G=2\Omega G$, where $\mathcal{L}_X$ denotes the Lie derivation with respect to X.

In the previous paper [1], we proved the following theorem.

Theorem. *Let M be an n-dimensional Riemannian manifold, and let $T(M)$ be its tangent bundle with the metric II. Then every infinitesiml fibre-preserving conformal transformation X on $T(M)$ naturally induces an infinitesimal projective transformation V on M. Furthermore the correspondence $X \to V$ gives a homomorphism of the Lie algebra of infinitesimal fibre-preserving conformal transformations of $T(M)$ onto the Lie algebra of infinitesimal projective transformations of M, and the kernel of this homomorphism is naturally homomorphic onto the Lie algebra of infinitesimal isometries of M.*

The purpose of the present paper is to prove the following theorem.

Theorem. *Let M be an n-dimensional Riemannian manifold, and let $T(M)$ be its tangent bundle with the metric $I+III$. Then every infinitesimal fibre-preserving conformal transformation X is a homothetic one and it induces an infinitesimal homothetic transformation V on M. Furthermore the correspondence $X \to V$ gives a homomorphism of the Lie algebra of infinitesimal fibre-preserving conformal transformations of $T(M)$ onto the Lie algebra of infinitesimal homothetic transformations of M, and the kernel of this homomorphism is naturally isomorphic onto the the Lie algebra of infinitesimal isometries of M.*

§ 1. Preliminaries.

Let $\Gamma_j{}^h{}_i$ be the coefficients of the Riemannian connection of M, then $y^a \Gamma_a{}^h{}_i$ can be regarded as coefficients of the non-linear connection of $T(M)$ where (x^h, y^h) the induced coordinates in $T(M)$. We put

$$X_h = \frac{\partial}{\partial x^h} - y^a \Gamma_a{}^m{}_h \frac{\partial}{\partial y^m} \qquad \text{and} \qquad X_{\bar{h}} = \frac{\partial}{\partial y^h},$$

then we call $\{X_h, X_{\bar{h}}\}$ the adapted frame of $T(M)$, and let $\{dx^h, \delta y^h\}$ be the dual basis of $\{X_h, X_{\bar{h}}\}$.

We can easily prove the following lemma.

Lemma 1. *The Lie brackets satisfy the following :*

$$[X_i, X_j] = y^r K_{jir}{}^m X_{\bar{m}},$$

$$[X_i, X_{\bar{j}}] = \Gamma_j{}^m{}_i X_{\bar{m}},$$

$$[X_{\bar{i}}, X_{\bar{j}}] = 0,$$

where $K_{jir}{}^m$ denote the components of the curvature tensor of M.

Let X be an infinitesimal fibre-preserving transformation on $T(M)$ and $(v^h, v^{\bar{h}})$ the components of X with respect to the adapted frame $\{X_h, X_{\bar{h}}\}$. Then X is fibre-preserving if and only if v^h depend only on the variables (x^h). Clearly X induces an infinitesimal transformation V with the components v^h in the base space M. Let $\pounds_X$ be the Lie derivation with respect to X, then we have the following lemma.

Lemma 2. (*See* [1]). *The Lie derivatives of the adapted frame and the dual basis are given as follows :*

(1) $\pounds_X X_h = -\partial_h v^a X_a + \{y^b v^c K_{hcb}{}^a - v^{\bar{b}} \Gamma_b{}^a{}_h - X_h(v^{\bar{a}})\} X_{\bar{a}}$.

(2) $\pounds_X X_{\bar{h}} = \{v^b \Gamma_b{}^a{}_h - X_{\bar{h}}(v^{\bar{a}})\} X_{\bar{a}}$,

(3) $\pounds_X dx^h = \partial_m v^h dx^m$,

(4) $\pounds_X \delta x^h = -\{y^b v^c K_{mcb}{}^h - v^{\bar{b}} \Gamma_b{}^h{}_m - X_m(v^{\bar{h}})\} dx^m - \{v^b \Gamma_b{}^h{}_m - X_{\bar{m}}(v^{\bar{h}})\} \delta y^m$.

Let g be a Riemannian metric of M with components g_{ji}, then we see that

$$\text{I} : G_{\text{I}} = g_{ji} dx^j dx^i,$$
$$\text{II} : G_{\text{II}} = 2 g_{ji} dx^j \delta y^i,$$
$$\text{III} : G_{\text{III}} = g_{ji} \delta y^j \delta y^i,$$

are all quadratic differential forms defined globally in $T(M)$ and that

$$\text{II} : 2 g_{ji} dx^j \delta y^i,$$
$$\text{I} + \text{II} : g_{ji} dx^j dx^i + 2 g_{ji} dx^j \delta y^i,$$
$$\text{I} + \text{III} : g_{ji} dx^j dx^i + g_{ji} \delta y^j \delta y^i,$$
$$\text{II} + \text{III} : 2 g_{ji} dx^j \delta y^i + g_{ji} \delta y^j \delta y^i,$$

are all non-singular and consequently can be regarded as Riemannian or pseudo-Riemannian metrics in $T(M)$.

Lemma 3. (*See* [1]). *The Lie derivatives $\pounds_X G_{\text{I}}$, $\pounds_X G_{\text{II}}$ and $\pounds_X G_{\text{III}}$ are given as follows :*

(1) $\pounds_X G_{\text{I}} = (\pounds_v g_{ij}) dx^i dx^j$,

(2) $1/2 \pounds_X G_{\text{II}} = -g_{im}\{y^b v^c K_{jcb}{}^m - v^{\bar{b}} \Gamma_b{}^m{}_j - X_j(v^{\bar{m}})\} dx^i dx^j$
$\qquad + \{\pounds_v g_{ij} - g_{im} \nabla_j v^m + g_{im} X_{\bar{j}}(v^{\bar{m}})\} dx^i \delta y^j$,

(3) $\pounds_X G_{\text{III}} = -2 g_{mj}\{y^b v^c K_{icb}{}^m - v^{\bar{b}} \Gamma_b{}^m{}_i - X_i(v^{\bar{m}})\} dx^i \delta y^j$
$\qquad + \{\pounds_v g_{ij} - 2 g_{mj} \nabla_i v^m + 2 g_{mj} X_{\bar{i}}(v^{\bar{m}})\} \delta y^i \delta y^j$,

where $\pounds_v g_{ij}$ denote the components of the Lie derivative $\pounds_v g$.

§ 2. Infinitesimal conformal transformations of the tangent bundles with the metric $G_{\mathrm{I+III}}$.

Let $T(M)$ be the tangent bundle with the metric $G_{\mathrm{I+III}}$, and let X be an infinitesimal fibre-preserving conformal transformation on $T(M)$, that is, there exists a scalar function Ω on $T(M)$ such that

$$\pounds_X G_{\mathrm{I+III}}=2\Omega G_{\mathrm{I+III}}.$$

Then from *Lemma 3*, we have

(2.1) $\pounds_v g_{ij}=2\Omega g_{ij}$,

(2.2) $y^b v^c K_{icb}{}^m - v^{\bar{b}}\, \Gamma_b{}^m{}_i - X_i(v^{\bar{m}})=0$,

and

(2.3) $g_{mj}(v^m{}_{|i}-X_{\bar{i}}(v^{\bar{m}}))+g_{mi}(v^m{}_{|j}-X_{\bar{j}}(v^{\bar{m}})=0$,

where $v^m{}_{|i}$ denote the components of the covariant derivative of V.

Thus by (2.1), the scalar function Ω on $T(M)$ can be regarded as a function on M. Hence the induced vector field V is an infinitesimal conformal transformation of M.

Proposition 1. *The components $v^{\bar{h}}$ of X can be written as the following form :*

(2.4) $$v^{\bar{h}}=y^a A^h{}_a+B^h$$

where $A^h{}_a$ and B^h are the components of a certain (1,1) tensor field and a certain contravariant vector field on M, respectively.

Proof. From (2.1) and (2.3), we get

(2.5) $$g_{mj}X_{\bar{i}}(v^{\bar{m}})+g_{mi}X_{\bar{j}}(v^{\bar{m}})=2\Omega g_{ij},$$

thus we have

$$g_{mi}X_{\bar{i}}(v^{\bar{m}})=\Omega g_{ii} \text{ and } g_{mj}X_{\bar{i}}X_{\bar{i}}(v^{\bar{m}})+g_{mi}X_{\bar{i}}X_{\bar{j}}(v^{\bar{m}})=0,$$

it follows that

$$X_{\bar{j}}X_{\bar{i}}(v^{\bar{m}})=0.$$

Hence $v^{\bar{h}}$ can be written as

$$v^{\bar{h}}=y^a A^h{}_a+B^h,$$

where $A^h{}_a$ and B^h are certain functions which depend only on the variables (x^h). Since X is a vector field on $T(M)$, we can easily show that $A^h{}_a$ and B^h are the components of a (1,1) tensor field and a contravariant vector field on M, respectively. *Q. E. D.*

Proposition 2. *The vector field B with the compnnents (B^h) is an infinitesimal isometry on M.*

Proof. Substituting (2.4) into (2.2), we have

(2.6) $$B^h{}_{|i}=0,$$

and

(2.7) $$A^h{}_{a|i}+K_{bia}{}^h v^b=0,$$

where $A^h{}_{a|i}$ denote the components of the covariant derivative of the (1,1) tensor field $A=(A^h{}_i)$. From (2.6), we get $\mathcal{L}_B g=0$, *this shows* B is an infinitesimal isometry on M. *Q. E. D.*

Proposition 3. *The scalar function* Ω *is constant.*

Proof. Substituting (2.4) into (2.5), we have

$$g_{mj}A^m{}_i+g_{mi}A^m{}_j=2\Omega\, g_{ji}$$

from which we get

$$g_{mj}A^m{}_{i|k}+g_{mi}A^m{}_{j|k}=2\Omega_k g_{ji},\ \textit{where}\ \Omega_k=\partial_k\Omega.$$

Substituting (2.7) into the above equation, we obtain $\Omega_k=0$. *Q. E. D.*

Now we consider the converse problem. Let M admits an infinitesimal homothetic transformation V with the components (v^h), that is,

$$\mathcal{L}_V g=2\Omega\, g \qquad (\Omega=constant).$$

We put

$$A_{ij}=\nabla_j v_i+\Omega\, g_{ij}-1/2\mathcal{L}_V g_{ij},$$

where $v_i=g_{im}v^m$, then by the Ricci identity, we get

(2.8) $$A_{ij|k}+K_{ijk}{}^m v_m=0.$$

Proposition 4. *The vector field* X *on* $T(M)$ *defined by*

$$X=v^h X_h+y^a A^h{}_a X_{\bar{h}}$$

is an infinitesimal fibre-preserving homothetic transformation on $T(M)$, *where* $A^h{}_i=g^{ha}A_{ai}$.

Proof. From (2.8), we can easily prove $\mathcal{L}_V G_{\mathrm{I+III}}=2\Omega\, G_{\mathrm{I+III}}$. *Q. E. D.*

Proof of Theorem. Summing up Proposition 1～Proposition 4, it is clear that the correspondence $X\to V$ gives a homomorphism of the Lie algebra of infinitesimal fibre-preserving conformal transformations of $T(M)$ onto the Lie algebra of infinitesimal homothetic transformations of M, and the kernel of this homomorphism is naturally isomorphic onto the Lie algebra of infisitesimal isometries of M.

Q. E. D.

References

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