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On infinitesimal conformal transformations of the tangent bundles over Riemannian manifolds

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Introduction. In the present paper everything will be always discussed in the C^∞ category, and Riemannian manifolds will be assumed to be connected and dimension>1.

Let M be a Riemannian manifold, and let ϕ be a transformation of M. Then ϕ is called a projective transformation, if it preserves the geodesics, where each geodesic should be confounded with a subset of M by neglecting its affine parameter. Furthermore ϕ is called an affine transformation, if it preserves the Riemannian connection. We may also speak of local projective and affine transformations. We then remark that a (local) affine transformation may be characterized as a (local) projective transformation which preserves the affine parameter together with the geodesics.

Let V be a vector field on M, and let us consider the local one-parameter group $\{\phi_t\}$ of local transformations of M generated by V. Then V is called an infinitesimal projective (resp. affine) transformation, if each ϕ_t is a local projective (resp. affine) transformation. By a complete infinitesimal projective transformation we mean an infinitesimal projective transformation which generates a (global) one-parameter group of projective transformations.

Clearly an infinitesimal affine transformation is an infinitesimal projective transformation. The converse is not true in general. Indeed consider the n-dimensional real projective space $P^n(R)$ with the standard Riemannian metric, which is the standard projectively flat Riemannian manifold, and is a space of positive constant curvature. As is well known, the Lie algebra of all infinitesimal projective transformations of $P^n(R)$ is isomorphic with the simple Lie algebra $\mathfrak{P}(n+1, R)$, and the Lie algebra of all infinitesimal affine transformations of $P^n(R)$ is isomorphic with the simple Lie algebra $\mathfrak{O}(n+1)$. In paticular it follows that, $P^n(R)$ admits a non-affine infinitesimal projective transformation (cf. [1]).

Many mathematicians have studied (infinitesimal) projective transformations of M and obtained interesting results. Above all, the following problem is famous:

"*Let M be a complete n-dimensional Riemannian manifold admitting a non-affine infinitesimal projective transformation. Then, is M a space of positive constant curvature*?"

As affirmative answers to this problem, see cf. [2], [3], [4], [5].

Let $T(M)$ be the tangent bundle over M, and let Φ be a transformation of $T(M)$. Then Φ is called a fibre-preserving transformation, if it preserves the fibres. Let X be a vector field on $T(M)$, and let us consider the local one-parameter group $\{\Phi_t\}$ of local transformations of $T(M)$ generated by X. Then X is called an infinitesimal fibre-preserving transformation on $T(M)$, if each Φ_t is a local fibre-preserving transformation of $T(M)$. Clearly an infinitesimal fibre-preserving transformation on $T(M)$ induces an infinitesimal transformation in the base space M. Let $\bar{g}$ be a Riemannian or pseudo-Riemannian metric of $T(M)$. An infinitesimal fibre-preserving transformation X on $T(M)$ is said to be an infinitesimal fibre-preserving conformal transformation, if there exists a scalar function $\bar{\rho}$ on $T(M)$ such that $\mathcal{L}_X\bar{g}=2\bar{\rho}\bar{g}$, where $\mathcal{L}_X$ denotes the Lie derivation with respect to X.

The main purpose of the present paper is to investigate some relations between the Lie algebra of infinitesimal fibre-preserving conformal transformations of the tangent bundle $T(M)$ and the Lie algebra of infinitesimal projective transformations of M. Then we shall prove the following theorem.

Theorem. *Let M be an n-dimensional Riemannian manifold, and let $T(M)$ be its tangent bundle with the metric II (see §2). Then every infinitesimal fibre-preserving conformal transformation X on $T(M)$ naturally induces an infinitesimal projective transformation V on M. Furthermore the corespondence $X\rightarrow V$ gives a homomorphism of the Lie algebra of infinitesimal fibre-preserving conformal transformations of $T(M)$ onto the Lie algebra of infinitesimal projective transformations of M, and the kernel of this homomorphism is naturaliy homomorphic onto the Lie* *algebra of infinitesimal isometries of M.*

§1. Preliminaries. In this section, we recall some fundamental facts on the tangent bundle $T(M)$ over M and on the non-linear connections of $T(M)$ for later use. Throughout the present paper, the indices $a, b, c, \ldots, h, i, j, \ldots$ run over the range $1, 2, \ldots$, n and the indices $A, B, C, \ldots, P, Q, R, \ldots$ run over the range $1, 2, \ldots, n, \bar{1}, \bar{2}, \ldots, \bar{n}$. The summation convention will be used with respect to this system of indices. Let π be the natural projection of $T(M)$ onto M and $\{U, x^h\}$ be a local coordinate system of M. Then each $\pi^{-1}(U)$ admits the induced coordinates (x^h, y^h). If $\{U', x^{h'}\}$ is another coordinate system of M and $U \cap U' \neq \phi$, then the induced coordinates $(x^{h'}, y^{h'})$ in $\pi^{-1}(U')$ are given by

$$x^{h'} = x^{h'}(x^h), \quad y^{h'} = \frac{\partial x^{h'}}{\partial x^h} y^h. \tag{1.1}$$

Putting $x^{\bar{h}} = y^h$, $x^{\bar{h}'} = y^{h'}$, we often write the equation (1. 1) as

$$x^{p'} = x^{p'}(x^A). \tag{1.2}$$

The Jacobian of (1. 2) is given by the matrix

$$\left(\frac{\partial x^{p'}}{\partial x^p}\right) = \begin{pmatrix} \dfrac{\partial x^{h'}}{\partial x^h} & 0 \\ \dfrac{\partial^2 x^{h'}}{\partial x^h \partial x^a} y^a & \dfrac{\partial x^{h'}}{\partial x^h} \end{pmatrix}$$

Hence the natural basis $\left(\frac{\partial}{\partial x^h}, \frac{\partial}{\partial y^h}\right)$ on $T(M)$ is transformed as follows

$$\frac{\partial}{\partial x^{h'}} = \frac{\partial x^h}{\partial x^{h'}} \frac{\partial}{\partial x^h} + \frac{\partial^2 x^h}{\partial x^{h'} \partial x^{a'}} y^{a'} \frac{\partial}{\partial y^h}, \tag{1.3}$$

$$\frac{\partial}{\partial y^{h'}} = \frac{\partial x^h}{\partial x^{h'}} \frac{\partial}{\partial y^h}.$$

Let N be a non-linear connection of $T(M)$, that is, N is a distribution

$$N : \bar{p} \in T(M) \to N_{\bar{p}} \subset T(T(M))_{\bar{p}},$$

such that

$$T(T(M))_{\bar{p}} = N_{\bar{p}} + T(T(M))^v{}_{\bar{p}} \quad \text{(direct sum)},$$

where $T(T(M))_{\bar{p}}$ is the tangent space of $T(M)$ at $\bar{p}$ and $T(T(M))^v{}_{\bar{p}}$ is the subspace of $T(T(M))_{\bar{p}}$ generated by $\frac{\partial}{\partial y^h}$, (h=1, 2, ..., n).

Functions $N^h{}_i$ on $T(M)$ which satisfy the following coordinate transformation rule are called coefficients of the non-linear connection N :

$$N^{h'}{}_{i'} = \frac{\partial x^{h'}}{\partial x^h}\left(\frac{\partial x^i}{\partial x^{i'}} N^h{}_i + \frac{\partial^2 x^h}{\partial x^{i'} \partial x^{a'}} y^{a'}\right). \tag{1.4}$$

Let $\Gamma_j{}^h{}_i$ be the coefficients of the Riemannian connection Γ of M, then $y^a \Gamma_a{}^h{}_i$ satisfy the transformation rule (1. 4), thus $y^a \Gamma_a{}^h{}_i$ can be regarded as coefficients of the non-linear connection N. If we put

$$\frac{\delta}{\delta x^h} = \frac{\partial}{\partial x^h} - y^a \Gamma_a{}^m{}_h \frac{\partial}{\partial y^m},$$

then we can easily prove $\left\{\frac{\delta}{\delta x^h}\right\}$ is a local basis of N by (1. 3), and we call $\left\{\frac{\delta}{\delta x^h}, \frac{\partial}{\partial y^h}\right\}$ the adapted frame of $T(M)$. In the following, we write $\{X_h, X_{\bar{h}}\}$ for the local basis $\left\{\frac{\delta}{\delta x^h}, \frac{\partial}{\partial y^h}\right\}$ for simplicity, and let $\{dx^h, \delta y^h\}$ be the dual basis of $\{X_h, X_{\bar{h}}\}$.

We can easily prove the following lemma.

Lemma 1. 1. *The Lie brackets satisfy the following* :

$$\begin{aligned} &[X_i, X_j] = y^r K_{jir}{}^m X_{\bar{m}}, \\ &[X_i, X_{\bar{j}}] = \Gamma_j{}^m{}_i X_{\bar{m}}, \\ &[X_{\bar{i}}, X_{\bar{j}}] = 0, \end{aligned} \tag{1.5}$$

where $K_{ijr}{}^m$ denote the components of the curvature tensor of M.

§2. Lie derivations. Let M be an n-dimensional Riemannian manifold. Let V be a vector field on M and let $\{\phi_t\}$ be any local group of local transformations of M generated by V. Take any tensor field S on M, and denote by $\tilde{\phi}_t(S)$ the pull-back of S by ϕ_t. Then the Lie derivative of S with respect to V is the tensor field $\mathcal{L}_V S$ on M defined by

$$\mathcal{L}_V S = \partial_t \tilde{\phi}_t(S)|_{t=0}, \quad \text{on the domain of } \phi_t.$$

The mapping $\mathcal{L}_V$ which maps S to $\mathcal{L}_V S$ is called the Lie derivation with respect to V.

Let us now give a local expression of the Lie derivative $\mathcal{L}_V S$ in terms of the (classical) covariant derivatives. For example suppose that S is a tensor field of type (1, 2). Then it can be shown that the components $\mathcal{L}_V S_j{}^h{}_i$ of $\mathcal{L}_V S$ my be expressed as follows:

$$\mathcal{L}_V S_j{}^h{}_i = v^a \nabla_a S_j{}^h{}_i - S_j{}^a{}_i \nabla_a v^h + S_a{}^h{}_i \nabla_j v^a + S_j{}^h{}_a \nabla_i v^a$$

where $S_j{}^h{}_i$ and v^h denote the components of S and V, and $\nabla_a S_j{}^h{}_i$ and $\nabla_a v^h$ the components of covariant derivatives of S and V, respectively.

Let g be a Riemannian metric of M with components g_{ij}, then V is called an infinitesimal isometry on M if it satisfies $\mathcal{L}_V g = 0$, that is,

$$\mathcal{L}_V g_{ij} = \nabla_j v_i + \nabla_i v_j = 0, \tag{2.1}$$

where $v_i = g_{ia} v^a$.

Next we define the Lie derivative of the Riemannian connection. Let V and $\{\phi_t\}$ be as above, and denote by $\tilde{\phi}_t(\Gamma)$ the bull-back of the Riemannian connection Γ on M by ϕ_t. Then the Lie derivative of Γ with respect to V is the tensor field $\mathcal{L}_V \Gamma$ of type (1, 2) on M defined by

$$\mathcal{L}_V \Gamma = \partial_t \tilde{\phi}_t(\Gamma)|_{t=0}, \quad \text{on the domain of } \phi_t.$$

It can be shown that the components $\mathcal{L}_V \Gamma_j{}^h{}_i$ of $\mathcal{L}_V \Gamma$ may be expressed as follows:

$$\mathcal{L}_V \Gamma_j{}^h{}_i = \nabla_j \nabla_i v^h + K_{aji}{}^h v^a. \tag{2.2}$$

Furthermore we can easily prove the following formula:

$$\mathcal{L}_V \Gamma_j{}^h{}_i = 1/2 g^{ha} [\nabla_j (\mathcal{L}_V g_{ai}) + \nabla_i (\mathcal{L}_V g_{ja}) - \nabla_a (\mathcal{L}_V g_{ji})]. \tag{2.3}$$

For an infinitesimal projective transformation V on M, we have the following well-known lemma.

Lemma 2.1. *A vector field V on M is an infinitesimal projective transformation if and only if there exists a covariant vector field ξ on M with the components ξ_i such that*

$$\mathcal{L}_V \Gamma_j{}^h{}_i = \delta^h{}_j \xi_i + \delta^h{}_i \xi_j.$$

Let X be an infinitesimal fibre-preserving transformation on $T(M)$ and $(v^h, v^{\bar{h}})$ the components of X with respect to the adapted frame $(X_h, X_{\bar{h}})$, that is,

$$X = v^h X_h + v^{\bar{h}} X_{\bar{h}}.$$

Then X is fibre-preserving, if and only if v^h depends only on the the variables x^1, ..., x^n with respect to the induced coordinates (x^h, y^h) in $T(M)$. Clearly X induces an infinitesimal transformation V with the components v^h in the base space M.

Let $\mathcal{L}_X$ be the Lie derivation with respect to the infinitesimal fibre-preserving transformation X on $T(M)$, then we have the following proposition.

Proposition 2. 2. *Let* $(X_h, X_{\bar{h}})$ *be the adapted frame on T(M) and* $(dx^h, \delta y^h)$ the dual basis of $(X_h, X_{\bar{h}})$. Then the Lie derivatives $\mathcal{L}_X X_h$, $\mathcal{L}_X X_{\bar{h}}$, $\mathcal{L}_X dx^h$ and $\mathcal{L}_X \delta y^h$ are given as follows:

(1) $\mathcal{L}_X X^h = -\partial_h v^a X_a + \{y^b v^c K_{hcb}{}^a - v^{\bar{b}} \Gamma_b{}^a{}_h - X_h(v^{\bar{a}})\} X_{\bar{a}}$,

(2) $\mathcal{L}_X X_{\bar{h}} = \{v^b \Gamma_h{}^a{}_b - X_{\bar{h}}(v^{\bar{a}})\} X_{\bar{a}}$,

(3) $\mathcal{L}_X dx^h = \partial_m v^h dx^m$,

(4) $\mathcal{L}_X \delta y^h = -\{y^b v^c K_{mcb}{}^h - v^{\bar{b}} \Gamma_b{}^h{}_m - X_m(v^{\bar{h}})\} dx^m - \{v^b \Gamma_m{}^h{}_b - X_{\bar{m}}(v^{\bar{h}})\} \delta y^m$.

Proof. By the definition of the Lie derivation $\mathcal{L}_X$, it follows that

$$\begin{aligned} \mathcal{L}_X X_h &= [X, X_h] \\ &= [v^a X_a + v^{\bar{a}} X_{\bar{a}}, X_h], \end{aligned}$$

thus by means of *Lemma 1. 1,* we get (1). By the same way we obtain (2). Next we prove (3). Since $(dx^h, \delta y^h)$ is the dual basis of $(X_h, X_{\bar{h}})$, we can put

$$\mathcal{L}_X dx^h = \alpha^h{}_m dx^m + \beta^h{}_m \delta y^m,$$

then from (1) and (2) of *Proposition 2. 2.*, it follows

$$\begin{aligned} 0 &= \mathcal{L}_X(dx^h(X_m)) \\ &= \alpha^h{}_m - \partial_m v^h, \\ 0 &= \mathcal{L}_X(dx^h(X_{\bar{m}})) \\ &= \beta^h{}_m. \end{aligned}$$

Thus we get (3). Similarly we have (4). *Q. E. D.*

Let g be a Riemannian metric of M with components g_{ji}, then we see that

I : $g_{\mathrm{I}} = g_{ji} dx^j dx^i$,

II : $g_{\mathrm{II}} = 2 g_{ji} dx^j \delta y^i$,

III : $g_{\mathrm{III}} = g_{ji} \delta y^j \delta y^i$,

are all quadratic differential forms defined globally in the tangent bundle $T(M)$ over M and that

$$\text{II} : 2g_{ji}dx^j\delta y^i,$$
$$\text{I}+\text{II} : g_{ji}dx^jdx^i+2g_{ji}dx^j\delta y^i,$$
$$\text{I}+\text{III} : g_{ji}dx^jdx^i+g_{ji}\delta y^j\delta y^i,$$
$$\text{II}+\text{III} : 2g_{ji}dx^j\delta y^i+g_{ji}\delta y^j\delta y^i,$$

are all non-singular and consequently can be regarded as Riemannian or pseudo-Riemannian metrics in the tangent bundle $T(M)$ over M.

Proposition 2.3. *The Lie derivatives* $\mathcal{L}_X g_{\text{I}}$, $\mathcal{L}_X g_{\text{II}}$, *and* $\mathcal{L}_X g_{\text{III}}$ *are given as follows* :

(1) $\mathcal{L}_X g_{\text{I}}=(\mathcal{L}_V g_{ji})\,dx^idx^j$

(2) $1/2\mathcal{L}_X g_{\text{II}}=-g_{im}\{y^bv^cK_{jcb}{}^m-v^{\bar b}\Gamma_b{}^m{}_j-X_j(v^{\bar m})\}dx^idx^j$
$\qquad +\{\mathcal{L}_V g_{ij}-g_{im}\nabla_j v^m+g_{im}X_{\bar j}(v^{\bar m})\}dx^i\delta y^j,$

(3) $\mathcal{L}_X g_{\text{III}}=-2g_{mj}\{y^bv^cK_{icb}{}^m-v^{\bar b}\Gamma_b{}^m{}_i-X_i(v^{\bar m})\}dx^i\delta y^j$
$\qquad +\{\mathcal{L}_V g_{ji}-2g_{mj}\nabla_i v^m+2g_{mj}X_{\bar i}(v^{\bar m})\}\delta y^i\delta y^j.$

Proof. By means of *Proposition 2. 2, we get*

$$\begin{aligned}\mathcal{L}_X g_{\text{I}}&=\mathcal{L}_X(g_{ji}dx^idx^j)\\&=X(g_{ji})\,dx^idx^j+2g_{ji}(\mathcal{L}_X dx^i)\,dx^j\\&=(\mathcal{L}_V g_{ij})\,dx^idx^j.\end{aligned}$$

Then it follows (1). By the same way we obtain (2) and (3). *Q. E. D.*

§3. Infinitesimal conformal transformations of the tangent bundles with the metric g_{II}.

Let $T(M)$ be the tangent bundle with the metric g_{II}, and let X be an infinitesimal fibre-preserving conformal transformation on $T(M)$, that is, there exsists a scalar function $\bar\rho$ on $T(M)$ such that

$$\mathcal{L}_X g_{\text{II}}=2\bar\rho g_{\text{II}}.$$

Then from (2) of *Proposition 2. 3*, we have

$$\mathcal{L}_V g_{ji}-g_{im}\nabla_j v^m+g_{im}X_{\bar j}(v^{\bar m})=2\bar\rho g_{ji}, \tag{3.1}$$

and

$$g_{im}\{y^bv^cK_{jcb}{}^m-v^{\bar b}\Gamma_b{}^m{}_j-X_j(v^{\bar m})\}+g_{jm}\{y^bv^cK_{icb}{}^m-v^{\bar b}\Gamma_b{}^m{}_i-X_i(v^{\bar m})\}=0. \tag{3.2}$$

Proposition 3.1. *The scalar function* $\bar\rho$ *on* $T(M)$ *depends only on the variables*

(x^h) *with respect to the induced coordinates* (x^h, y^h).

Proof. Applying $X_{\bar{k}}$ to the both sides of the equation (3. 1), then we have

$$g_{im} X_{\bar{k}} X_{\bar{j}} (v^{\bar{m}}) = 2 X_{\bar{k}} (\bar{\rho}) g_{ij},$$

from which we get

$$X_{\bar{k}} (\bar{\rho}) g_{ji} = X_{\bar{j}} (\bar{\rho}) g_{ik},$$

it follows that

$$(n-1) X_{\bar{k}} (\bar{\rho}) = 0.$$

This means the scalar function $\bar{\rho}$ on T(M) depends only on the variables (x^h) with respect to the induced coordinates (x^h, y^h). *Q. E. D.*

Thus we can regard $\bar{\rho}$ is a function on M, in the following we write ρ insted of $\bar{\rho}$.

From (3. 1) and *Proposition 3. 1*, $X_{\bar{j}} (v^{\bar{m}})$ depends only on the variables (x^h) thus we can put

$$v^{\bar{m}} = y^a A^m{}_a + B^m, \tag{3.3}$$

where $A^m{}_a$ and B^m are certain functions which depend only on the variables (x^h). Furthermore we can easily show that $A^m{}_a$ and B^m are the components of a (1, 1) tensor field and a contravariant vector field on M, respectively.

Proposition 3. 2. *If we put*

$$B = B^b \frac{\partial}{\partial x^b},$$

then the vector field B on M is an infinitesimal isometry of M.

Proof. Substituting (3. 3) into (3. 2), we have

$$\nabla_j B_i + \nabla_i B_j = 0, \tag{3.4}$$

and

$$v^a (K_{jahi} + K_{iahj}) - \nabla_j A_{ih} - \nabla_i A_{jh} = 0. \tag{3.5}$$

where $B_i = g_{im} B^m$ and $A_{ih} = g_{im} A^m{}_h$.

Hence by (3. 4), it follows

$$\pounds_B g_{ji} = \nabla_j B_i + \nabla_i B_j = 0.$$

This shows B is an infinitesimal isometry on M from (2. 1). *Q. E. D.*

Proposition 3. 3. *If we put*

$$V=v^h\frac{\partial}{\partial x^h},$$

then the vector field V on M is an infinitesimal projective transformation of M.

Proof. Substituting (3.3) into (3. 1), it follows

$$A_{ij}=\nabla_j v_i+2\rho g_{ij}-\mathcal{L}_V g_{ij}. \tag{3.6}$$

Substituting (3. 6) into (3. 5), we obtain

$$\mathcal{L}_V\Gamma_i{}^h{}_j=\delta^h{}_i\rho_j+\delta^h{}_j\rho_i,$$

where $\rho_i=\nabla_i\rho$. This shows V is an infinitesimal projective transformation on M from *Lemma 2. 1.* *Q. E. D.*

Now we consider the converse problem, that is, let M admits an infinitesimal projective transformation $V=v^h\frac{\partial}{\partial x^h}$. Then we have the following proposition.

Proposition 3. 4. *The vector field X on T(M) defined by*

$$X=v^hX_h+y^aA^h{}_aX_{\bar{h}}$$

is an infinitesimal fibre-preserving conformal transformation on T(M), where

$A^h{}_i=g^{ha}A_{ai}$, $A_{ij}=\nabla_j v_i+2\rho g_{ij}-\mathcal{L}_V g_{ij}$ *and* $\rho=\frac{1}{n+1}\nabla_a v^a$.

Proof. By *Proposition 2. 2*, it follows that

$$\begin{aligned}\mathcal{L}_X g_{II}&=\mathcal{L}_X(2g_{ij}dx^i\delta y^j)\\&=2X(g_{ij})\,dx^i\delta y^j+2g_{ij}(\mathcal{L}_X dx^i)\,\delta y^j+2g_{ij}dx^i(\mathcal{L}_X\delta y^j)\\&=4\rho g_{ij}dx^i\delta y^j+2y^a(v^bK_{bjai}+\nabla_j A_{ia})\,dx^idx^j.\end{aligned}$$

On the other hand, by means of (2. 2), (2. 3) and (3. 6) we have

$$\begin{aligned}\nabla_j A_{ia}&=g_{im}\nabla_j\nabla_a v^h+2\rho_j g_{ia}-(\mathcal{L}_V\Gamma_j{}^m{}_i)\,g_{ma}-(\mathcal{L}_V\Gamma_j{}^m{}_a)\,g_{im}\\&=-v^bK_{bjai}+\rho_i g_{ia}-\rho_i g_{ja},\end{aligned}$$

from which we obtain

$$\mathcal{L}_X g_{II}=2\rho g_{II}.$$

Hence X is an infinitesimal fibre-preserving conformal transformation on $T(M)$. *Q. E. D.*

Proof of Theorem. Let M be an n-dimensional Riemannian manifold, and let $T(M)$ be its tangent bundle with the metric g_{II}. Then summing up *Proposition 3. 1*

$\sim$*3. 4*, it is clear that the correspondence $X \to V$ gives a homomorphism of the Lie algebra of infinitesimal fibre-preserving conformal transformations of $T(M)$ onto the Lie algebra of infinitesimal projectiove transformations of M, and the kernel of this homomorphism is naturally homomorphic onto the Lie algebra of infinitesimal isometries of M. *Q. E. D.*

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