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On Scalar Curvatures of Tangent Bundles of Riemannian Manifolds

Kazunari YAMAUCHI

Introduction.

Let M be an n-dimensional Riemannian manifold and $T(M)$ be its tangent bundle. It is well known that there exist many Riemannian or pseudo-Riemannian metrics in $T(M)$, which are defined by *K. Yano* and *S. Ishihara*. ([1]). Concerning these metrics, they showed following theorem.

Theorem 1. *Let M be an n-dimensional Riemannian manifold and T(M) be its tangent bundle with the metric* Ⅱ *or* Ⅰ＋Ⅱ, *then the scalar curvatur of T(M) vanishes.*

The purpose of the present paper is to calculate the scalar curvatures of $T(M)$ with the metric Ⅰ＋Ⅲ or Ⅱ＋Ⅲ, and prove the following theorem.

Theorem 2. *Let M be an n-dimensional Riemannian manifold and T(M) be its tangent bundle with the metric* Ⅰ＋Ⅲ or Ⅱ＋Ⅲ. *If the scalar curvature of T(M) is constant, then M is a locally flat space.*

For the case with the metric Ⅰ＋Ⅲ, see [2].

§1. Preliminaries.

Let π be the natural projection of $T(M)$ onto M and $\{(U, x^h)\}$ be a local coordinate system of M. Then each $\pi^{-1}(U)$ admits the induced coordinates (x^h, y^h). Putting $x^{\bar{h}}=y^h$, we often write (x^h, y^h) as (x^A). The indices $a, b, c, \ldots, h, i, j, \ldots$ run over the range $1, 2, 3, \ldots, n$ and $A, B, C, \ldots, P, Q, R, \ldots$ run over the range $1, 2, 3, \ldots, n, \bar{1}, \bar{2}, \bar{3}, \ldots, \bar{n}$. The summation convention will be used with respect to this system of indices.

1. 1. Non-linear connections. Let N be a non-linear connection of $T(M)$, that is, N is a distribution $N : \bar{p} \in T(M) \longrightarrow N_{\bar{p}} \subset T(T(M))_{\bar{p}}$ such that

$$T(T(M))_{\bar{p}} = N_{\bar{p}} + T(T(M))^{v}, \quad \text{(direct sum)},$$

where $T(T(M))_{\bar{p}}$ is a tangent space of $T(M)$ at $\bar{p}$ and $T(T(M))^{v}$ is a subspace of $T(T(M))_{\bar{p}}$ generated by $\partial/\partial y^{h}$ $(h=1, 2, \ldots, n)$. Functions $N^{h}{}_{i}$ on $T(M)$ which satisfy the following coordinates transformation rule are called coefficients of the non-linear connection N:

$$N^{h'}{}_{i'} = \frac{\partial x^{h'}}{\partial x^{h}} \left(\frac{\partial x^{i}}{\partial x^{i'}} N^{h}{}_{i} + \frac{\partial^{2} x^{h}}{\partial x^{i'} \partial x^{a'}} y^{a'} \right). \tag{1. 1}$$

Let $\Gamma_{j}{}^{h}{}_{i}$ be the coefficients of a Riemannian connection of M, then

$$y^{a} \Gamma_{a}{}^{h}{}_{i} \tag{1. 2}$$

satisfy the transformation rule (1. 1), thus, in the following, we use (1. 2) as the coefficients of the non-linear connection of $T(M)$. We put

$$\frac{\delta}{\delta x^{i}} = \frac{\partial}{\partial x^{i}} - N^{h}{}_{i} \frac{\partial}{\partial y^{h}}, \tag{1. 3}$$

then we can easily prove $\{\delta/\delta x^{i} \mid i=1, 2, \ldots, n\}$ is a local basis of $N_{\bar{p}}$, and we call $(\delta/\delta x^{h}, \partial/\partial y^{h})$ the N frame of $T(M)$. In the following, we write $(X_{h}, X_{\bar{h}})$ for the local basis $(\delta/\delta x^{h}, \partial/\partial y^{h})$ of $T(T(M))$ for simplicity and let $(dx^{h}, \delta y^{h})$ be the dual basis of $(X_{h}, X_{\bar{h}})$.

1. 2. Linear connections. Let $\overline{\nabla}$ be a linear connection on $T(M)$ and $\overline{\Gamma}_{B}{}^{A}{}_{C}$ be the coefficients of $\overline{\nabla}$, that is, $\overline{\Gamma}_{B}{}^{A}{}_{C}$ satisfy the following

$$\begin{aligned} \overline{\nabla}_{X_{i}} X_{j} &= \overline{\Gamma}_{j}{}^{m}{}_{i} X_{m} + \overline{\Gamma}_{j}{}^{\bar{m}}{}_{i} X_{\bar{m}}, \\ \overline{\nabla}_{X_{i}} X_{\bar{j}} &= \overline{\Gamma}_{\bar{j}}{}^{m}{}_{i} X_{m} + \overline{\Gamma}_{\bar{j}}{}^{\bar{m}}{}_{i} X_{\bar{m}}, \\ \overline{\nabla}_{X_{\bar{i}}} X_{j} &= \overline{\Gamma}_{j}{}^{m}{}_{\bar{i}} X_{m} + \overline{\Gamma}_{j}{}^{\bar{m}}{}_{\bar{i}} X_{\bar{m}}, \\ \overline{\nabla}_{X_{\bar{i}}} X_{\bar{j}} &= \overline{\Gamma}_{\bar{j}}{}^{m}{}_{\bar{i}} X_{m} + \overline{\Gamma}_{\bar{j}}{}^{\bar{m}}{}_{\bar{i}} X_{\bar{m}}, \end{aligned} \tag{1. 4}$$

from which, we have

$$\begin{aligned} \overline{\nabla}_{X_{i}} dx^{h} &= -\overline{\Gamma}_{m}{}^{h}{}_{i} dx^{m} - \overline{\Gamma}_{\bar{m}}{}^{h}{}_{i} \delta y^{m}, \\ \overline{\nabla}_{X_{i}} \delta y^{h} &= -\overline{\Gamma}_{m}{}^{\bar{h}}{}_{i} dx^{m} - \overline{\Gamma}_{\bar{m}}{}^{\bar{h}}{}_{i} \delta y^{m}, \\ \overline{\nabla}_{X_{\bar{i}}} dx^{h} &= -\overline{\Gamma}_{m}{}^{h}{}_{\bar{i}} dx^{m} - \overline{\Gamma}_{\bar{m}}{}^{h}{}_{\bar{i}} \delta y^{m}, \\ \overline{\nabla}_{X_{\bar{i}}} \delta y^{h} &= -\overline{\Gamma}_{m}{}^{\bar{h}}{}_{\bar{i}} dx^{m} - \overline{\Gamma}_{\bar{m}}{}^{\bar{h}}{}_{\bar{i}} \delta y^{m}. \end{aligned} \tag{1. 5}$$

According to (1. 1) and (1. 2), the Lie brackets satisfy the following

$$\begin{aligned} [X_{i}, X_{j}] &= y^{r} K_{jir}{}^{m} X_{\bar{m}}, \\ [X_{i}, X_{\bar{j}}] &= \Gamma_{j}{}^{m}{}_{i} X_{\bar{m}}, \end{aligned} \tag{1. 6}$$

$$[X_{\bar{i}},\ X_{\bar{j}}]=0,$$

where $K_{jir}{}^{m}$ denote the components of the curvature tensor of M and $\Gamma_j{}^m{}_i$ the coefficients of a Riemannian connection of M.

1\. 3. Riemannian or pseudo-Riemannian metrics of $T(M)$. Let g_{ji} be the components of a Riemannian metric of M, then we see that

$$\begin{aligned} \mathrm{I} &: g_{ji}dx^j dx^i, \\ \mathrm{II} &: 2g_{ji}dx^j \delta y^i, \\ \mathrm{III} &: g_{ji}\delta y^j \delta y^i, \end{aligned}$$

are all quadratic differential forms defined globally in the tangent bundle $T(M)$ and that

$$\begin{aligned} \mathrm{II} &: 2g_{ji}dx^j \delta y^i, \\ \mathrm{I}+\mathrm{II} &: g_{ji}dx^j dx^i + 2g_{ji}dx^j \delta y^i, \\ \mathrm{I}+\mathrm{III} &: g_{ji}dx^j dx^i + g_{ji}\delta y^j \delta y^i, \\ \mathrm{II}+\mathrm{III} &: 2g_{ji}dx^j \delta y^i + g_{ji}\delta y^j \delta y^i, \end{aligned}$$

are all non-singular and consequently can be regarded as Riemannian or pseudo-Riemannian metrics in the tangent bundle $T(M)$.

§2. The coefficients $\overline{\Gamma_B{}^A{}_C}$ of $\overline{\nabla}$.

Let $\bar{g}$ be the Riemannian or pseudo-Riemannian metric defined above and $\overline{\nabla}$ be the Levi-Civita connection of $T(M)$, that is, $\overline{\nabla}$ satisfies

(2\. 1) $\overline{\nabla}_X \bar{g}=0$, for $\forall X \in T(T(M))$,

and

(2\. 2) $T(X,\ Y)=\overline{\nabla}_X Y-\overline{\nabla}_Y X-[X,\ Y]=0$, for $\forall X,\ Y \in T(T(M))$,

where $T(X,\ Y)$ denotes the torsion tensor of $T(M)$.

From *(1. 4)*, *(1. 6)* and *(2. 2)*, the coefficients $\overline{\Gamma_B{}^A{}_C}$ of the Levi-Civita connection $\overline{\nabla}$ satisfy the following

$$\text{(2. 3)}\qquad \begin{aligned} &\overline{\Gamma_j{}^h{}_i}=\overline{\Gamma_i{}^h{}_j}, &&\overline{\Gamma_j{}^{\bar{h}}{}_i}=\overline{\Gamma_i{}^{\bar{h}}{}_j}+y^r K_{jir}{}^h, \\ &\overline{\Gamma_{\bar{j}}{}^h{}_i}=\overline{\Gamma_i{}^h{}_{\bar{j}}}, &&\overline{\Gamma_{\bar{j}}{}^{\bar{h}}{}_i}=\overline{\Gamma_i{}^{\bar{h}}{}_{\bar{j}}}+\Gamma_j{}^h{}_i, \\ &\overline{\Gamma_{\bar{j}}{}^h{}_{\bar{i}}}=\overline{\Gamma_{\bar{i}}{}^h{}_{\bar{j}}}, &&\overline{\Gamma_{\bar{j}}{}^{\bar{h}}{}_{\bar{i}}}=\overline{\Gamma_{\bar{i}}{}^{\bar{h}}{}_{\bar{j}}}. \end{aligned}$$

Then from *(1. 5)*, *(2. 1)* and *(2. 3)*, we have the following propositions.

Proposition 2. 1. *Let M be an n-dimensional Riemannian manifold and $T(M)$ be its tangent bundle with the metric* II, *then the coefficients* $\overline{\Gamma_B{}^A{}_C}$ *satisfy*

$$\begin{aligned} &\overline{\Gamma_j{}^h{}_i}=\Gamma_j{}^h{}_i, &&\overline{\Gamma_j{}^{\bar{h}}{}_i}=y^r K_{rij}{}^h, \\ &\overline{\Gamma_{\bar{j}}{}^h{}_i}=0, &&\overline{\Gamma_j{}^h{}_{\bar{i}}}=0, \end{aligned}$$

$$\overline{\Gamma}_{\bar{j}}{}^{\bar{h}}{}_{i}=\Gamma_{j}{}^{h}{}_{i},\qquad \overline{\Gamma}_{j}{}^{\bar{h}}{}_{\bar{i}}=0,$$
$$\overline{\Gamma}_{\bar{j}}{}^{h}{}_{\bar{i}}=0,\qquad \overline{\Gamma}_{\bar{j}}{}^{\bar{h}}{}_{\bar{i}}=0.$$

Proposition 2. 2. *Let M be an n-dimensional Riemannian manifold and T(M) be its tangent bundle with the metric* Ⅰ+Ⅱ, *then the coefficients* $\overline{\Gamma}_{B}{}^{A}{}_{C}$ *satisfy*

$$\overline{\Gamma}_{j}{}^{h}{}_{i}=\Gamma_{j}{}^{h}{}_{i},\qquad \overline{\Gamma}_{j}{}^{\bar{h}}{}_{i}=y^{r}K_{rij}{}^{h},$$
$$\overline{\Gamma}_{\bar{j}}{}^{h}{}_{i}=0,\qquad \overline{\Gamma}_{j}{}^{h}{}_{\bar{i}}=0,$$
$$\overline{\Gamma}_{\bar{j}}{}^{\bar{h}}{}_{i}=\Gamma_{j}{}^{h}{}_{i},\qquad \overline{\Gamma}_{j}{}^{\bar{h}}{}_{\bar{i}}=0,$$
$$\overline{\Gamma}_{\bar{j}}{}^{h}{}_{\bar{i}}=0,\qquad \overline{\Gamma}_{\bar{j}}{}^{\bar{h}}{}_{\bar{i}}=0.$$

Proposition 2. 3. *Let M be an n-dimensional Riemannian manifold and T(M) be its tangent bundle with the metric* Ⅰ+Ⅲ, *then the coefficients* $\overline{\Gamma}_{B}{}^{A}{}_{C}$ *satisfy*

$$\overline{\Gamma}_{j}{}^{h}{}_{i}=\Gamma_{j}{}^{h}{}_{i},\qquad \overline{\Gamma}_{j}{}^{\bar{h}}{}_{i}=\tfrac{1}{2}y^{r}K_{jir}{}^{h},$$
$$\overline{\Gamma}_{\bar{j}}{}^{h}{}_{i}=\tfrac{1}{2}y^{r}K_{rji}{}^{h},\qquad \overline{\Gamma}_{j}{}^{h}{}_{\bar{i}}=\tfrac{1}{2}y^{r}K_{rij}{}^{h},$$
$$\overline{\Gamma}_{\bar{j}}{}^{\bar{h}}{}_{i}=\Gamma_{j}{}^{h}{}_{i},\qquad \overline{\Gamma}_{j}{}^{\bar{h}}{}_{\bar{i}}=0,$$
$$\overline{\Gamma}_{\bar{j}}{}^{h}{}_{\bar{i}}=0,\qquad \overline{\Gamma}_{\bar{j}}{}^{\bar{h}}{}_{\bar{i}}=0.$$

Proposition 2. 4. *Let M be an n-dimensional Riemannian manifold and T(M) be its tangent bundle with the metric* Ⅱ+Ⅲ, *then the coefficients* $\overline{\Gamma}_{B}{}^{A}{}_{C}$ *satisfy*

$$\overline{\Gamma}_{j}{}^{h}{}_{i}=\Gamma_{j}{}^{h}{}_{i}-\tfrac{1}{2}y^{r}(K_{rij}{}^{h}+K_{rji}{}^{h}),$$
$$\overline{\Gamma}_{j}{}^{\bar{h}}{}_{i}=y^{r}K_{rij}{}^{h},$$
$$\overline{\Gamma}_{\bar{j}}{}^{h}{}_{i}=-\tfrac{1}{2}y^{r}K_{rji}{}^{h},\qquad \overline{\Gamma}_{j}{}^{h}{}_{\bar{i}}=-\tfrac{1}{2}y^{r}K_{rij}{}^{h},$$
$$\overline{\Gamma}_{\bar{j}}{}^{\bar{h}}{}_{i}=\Gamma_{j}{}^{h}{}_{i}+\tfrac{1}{2}y^{r}K_{rji}{}^{h},\qquad \overline{\Gamma}_{j}{}^{\bar{h}}{}_{\bar{i}}=\tfrac{1}{2}y^{r}K_{rij}{}^{h},$$
$$\overline{\Gamma}_{\bar{j}}{}^{h}{}_{\bar{i}}=0,\qquad \overline{\Gamma}_{\bar{j}}{}^{\bar{h}}{}_{\bar{i}}=0.$$

§3. The scalar curvatures of T(M).

The curvature tensor $\bar{K}$ of $T(M)$ is defined by

$$\bar{K}(X,Y)Z=\overline{\nabla}_{X}\overline{\nabla}_{Y}Z-\overline{\nabla}_{Y}\overline{\nabla}_{X}Z-\overline{\nabla}_{[X,Y]}Z$$

and let $\bar{g}_{AB}$ be the components of $\bar{g}$ and $\bar{K}_{ABCD}$ the components of $\bar{K}$, that is,

$$\bar{K}_{ABCD}=\bar{g}(\bar{K}(X_{A},\ X_{B})X_{C},\ X_{D}).$$

Then the scalar curvatur $\bar{S}$ of $T(M)$ is given by

$$\bar{S}=\bar{g}^{AB}\bar{K}_{CAB}{}^{C}$$

where $\bar{g}^{AB}$ denote the components of the inverse matrix of $(\bar{g}_{AB})$ and $\bar{K}_{CAB}{}^{C}=\bar{g}^{CD}\bar{K}_{CABD}$.

By means of Proposition 2. 1 or Proposition 2. 2, we can easily prove Theorem 1.

From Proposition 2. 3 or Proposition 2. 4, we can prove the following propositions.

Proposition 3. 1. *Let M be an n-dimensional Riemannian manifold and T(M) be its tangent bundle with the metric* Ⅰ+Ⅲ, *then the scalar curvature* $\bar{S}$ *of T(M) is given by*

$$\bar{S}=S-\tfrac{1}{4}\, y^r y^s K_{rabc} K_s{}^{abc}$$

where S denotes the scalar curvature of M.

Proposition 3. 2. *Let M be an n-dimensional Riemannian manifold and T(M) be its tangent bundle with the metric* Ⅱ+Ⅲ, *then the scalar curvature* $\bar{S}$ *of T(M) is given by*

$$\bar{S}=-\tfrac{1}{2}\, S+3/8\; y^r y^s K_{rabc} K_s{}^{abc}$$

where S denotes the scalar curvature of M.

From Proposition 3. 1 or Proposition 3. 2, it follows that

Theorem 2. *Let M be an n-dimensional Riemannian manifold and T(M) be its tangent bundle with the metric* Ⅰ+Ⅲ or Ⅱ+Ⅲ. *If the scalar curvature of T(M) is constant, then M is a locally flat space.*

References

[1] K. Yano & S. Ishihara : Tangent and Cotangent Bundles, Marcel Dekker, Inc. 1973.

[2] K. Yamauchi : A Remark on the Curvature Tensors in Tangent Bundles, Science Reports of Kagoshima University, No. 35 (1986) 91-94.

Department of Mathematics
Asahikawa Medical College
Asahikawa, Japan