Ann. Rep. Asahikawa Med. Coll.
1991. Vol.12, 1~11

A Study of Special Subspaces in a Finsler Space, II

Dedicated to Professor Dr. Makoto Matsumoto On his seventieth birth day

Hiroshi YASUDA and Vahid ALLI*)

Introduction. This paper is a continuation of the previous papers ([4],[1] [8]). In Riemannian geometry, the following theorem is well known : If an n-dimensional Riemannian space M_n is of constant curvature and an m-dimensional subspace M_m of M_n is totally goedesic or totally umbilical, then M_n is also of constant curvature.

When, in particular, $m = n\text{-}1$, that is, the subspace is a hypersurface of M_n, we, in the paper [4], discussed in detail under the TM (or $TM(o)$)-connections how the above theorem is generalized to Finsler geometry. In the case where the codimension $(n\text{-}m)$ is more than 1, we, in the paper [8], studied the same problem under the T (or $T(o)$)-connections, which are the more general ones.

The principal purpose of the present paper is to develop the theory presented in the above paper [8]. The terminology and notations refer to papers [4]~[8] unless otherwise stated.

---

*) This author's university is shown at the end of the paper.

1) Numbers in brackets refer to the references at the end of the paper.

§1. Preliminaries. Let M_n be an n-dimensional Finsler space with a fundamental function $L(x^i, y^i)$ and be endowed with a Matsumoto connection $M\Gamma = (\Gamma_j{}^i{}_k, \Gamma^i{}_k, \widetilde{C}_j{}^i{}_k)$. Then it follows ([5], [6]) that $\widetilde{C}_j{}^i{}_k$ is a $(-1)p$-homogeneous tensor and we have

$$(1.1)\qquad \Gamma^i{}_k = G^i{}_k + T^i{}_k,\ \Gamma_j{}^i{}_k = \Gamma^i{}_{k\|j} + Q_j{}^i{}_k = G_j{}^i{}_k + T_j{}^i{}_k + Q_j{}^i{}_k,$$

where the symbol $\|j$ indicates the partial differentiation by y^j, $G^i{}_k$ and $G_j{}^i{}_k$ are the non-linear connection and h-connection of Berwald and $T^i{}_k$ and $Q_j{}^i{}_k$ are $(1)p$- and $(0)p$-homogeneous[2] tensors respectively.

Now we consider an m-dimensional subspace M_m of M_n defined by

$$(1.2)\qquad x^i = x^i(u^\alpha)\ (i=1, 2, \cdots\cdots, n : \alpha=1, 2. \cdots\cdots\cdots . m),$$

where variables u^α form a coordinate system of M_m and the matrix with components $B^i{}_\alpha\ (=\partial x^i/\partial u^\alpha)$ is of rank m.

If we denote the components of a vector y^i tangent to a curve in M_m by y^α[3] in terms of u^α-system, then we have

$$(1.3)\qquad y^i = B^i{}_\beta\, y^\beta,\ y^i{}_{\|\gamma}\ (= \partial y^i/\partial y^\gamma) = B^i{}_\gamma.$$

We choose $(n\text{-}m)$ normal vector N^i_b $(b = m+1, \cdots\cdots, n)$ at each point (u^α) of M_m such that

$$(1.4)\qquad g_{ij}\, N^i_b\, N^j_c = \delta_{bc},\ B^i{}_\gamma\, N^b_i = 0,\ N^b_i := g_{ij}\, N^j_b,$$

where g_{ij} is the metric tensor on M_n.

Further we have

$$\bar{L} = L\ (x^i(u^\alpha), B^i{}_\beta y^\beta),\ g_{\beta\gamma} = g_{jk} B^{jk}_{\beta\gamma},\ B^{jk}_{\beta\gamma} := B^j{}_\beta B^k{}_\gamma,$$

$$(1.5)\qquad B_i{}^\beta{}_{\|g} = 2C_b{}^\beta{}_\gamma N^b_i,\quad N^i_{b\,\|\gamma} = -2C_b{}^\beta{}_\gamma B^i{}_\beta - \lambda^c_{b\gamma}\, N^i_c,$$

$$N^b_{i\,\|\gamma} = \lambda^b_{c\gamma}\, N^c_i,\quad \lambda^b_{c\gamma} + \lambda^c_{b\gamma} = 2C_b{}^c{}_\gamma = 2C_c{}^b{}_\gamma,\ C_b{}^\beta{}_\gamma = C_j{}^i{}_k B_i{}^\beta B^k{}_\gamma N^j_b$$

2) "$(r)p$-homogeneous" means "positively homogeneous of degree r in y^i.

3) If no confusion occurs then we use y^α in stead of the usual notation v^α.

where L and $g_{\beta\gamma}$ are the induced fundamental function and metric tensor on M_m respectively and

(1. 6) $\quad C_{ijk} = \frac{1}{2} g_{ij\|k},\ C_j{}^i{}_k = C_{jsk} g^{si},\ B_i{}^\beta = g_{ij} B^j{}_\gamma g^{\gamma\beta},\ \lambda^a_{b\gamma} = -N^a_i N^i_{b\|\gamma},$

g^{jk} and $g^{\beta\gamma}$ being the reciprocal tensors of g_{jk} and $g_{\beta\gamma}$ respectively.

The included Matsumoto connection $IM\Gamma = (\Gamma_\beta{}^\alpha{}_\gamma, \Gamma^\alpha{}_\gamma, \widetilde{C}_\beta{}^\alpha{}_\gamma)$ on M_m is given by [5]

(1. 7)
$$\Gamma_\beta{}^\alpha{}_\gamma = B_i{}^\alpha (B_\beta{}^i{}_\gamma + \Gamma_j{}^i{}_k B^{jk}_{\beta\gamma}) + \widetilde{C}_\beta{}^\alpha{}_b H^b_\gamma,$$
$$\Gamma^\alpha{}_\gamma = B_i{}^\alpha (B_o{}^i{}_\gamma + \Gamma^i{}_k B^k{}_\gamma),\ \widetilde{C}_\beta{}^\alpha{}_\gamma = \widetilde{C}_j{}^i{}_k B_i{}^\alpha B^{jk}_{\beta\gamma},$$

where $B_\beta{}^i{}_\gamma = \partial B^i{}_\beta / \partial u^\gamma$, $B_o{}^i{}_\gamma = B_\beta{}^i{}_\gamma y^\beta$ and $C_\beta{}^\alpha{}_b = C_j{}^i{}_k B_i{}^\alpha B^j{}_\beta N^k_b$.

The normal curvature vector H^b_γ and the second fundamental tensor $H_\beta{}^b{}_\gamma$ in a direction N^i_b are given by

(1. 8) $\quad H^b_\gamma = N^b_i (B_o{}^i{}_\gamma + \Gamma^i{}_k B^k{}_\gamma).$

(1. 9) $\quad H_\beta{}^b{}_\gamma = N^b_i (B_\beta{}^i{}_\gamma + \Gamma_j{}^i{}_k B^{jk}_{\beta\gamma}) + \widetilde{C}_\beta{}^b{}_c,\ \widetilde{C}_\beta{}^b{}_c = \widetilde{C}_j{}^i{}_k B^j{}_\beta N^k_c N^b_i.$

The h-curvature tensor $R_{\alpha\delta\beta\gamma}$ with respect to $IM\Gamma$ *is given by*

(1. 10)
$$R_{\alpha\delta\beta\gamma} = R_{jikh} B^{ji}_{\alpha\delta} B^{kh}_{\beta\gamma} + B^{ji}_{\alpha\delta} \{ P_{jikh} (B^k{}_\beta H^b_\gamma - B^k{}_\gamma H^b_\beta) N^h_b$$
$$+ S_{jikh} N^k_b N^h_c H^b_\beta H^h_\gamma \} + [H_\alpha{}^b{}_\beta (g_{jk|\gamma} B^j{}_\delta N^k_b + \delta_{bc} H_\delta{}^c{}_o) - \beta|\gamma],$$

where R_{jikh}, P_{jikh}, S_{jikh} are the h-, hv-, v-curvature tensors with respect to $M\Gamma$ *respectively and the symbol* $\beta|\gamma$ means the interchange of indices β and γ in the foregoing terms within brackets.

If we contract (1. 10) by $y^\alpha y^\beta$ then we have

(1. 11)
$$R_{o\delta o\gamma} = R_{oioh} B^{ih}_{\delta\gamma} + S_{oikh} B^i{}_\delta N^k_b N^h_c H^b_o H^c_\gamma + B^i{}_\delta N^h_b (P_{oioh} H^b_\gamma - P_{oikh} B^k{}_\gamma H^b_o)$$
$$+ H_o{}^b{}_o (g_{jk|\gamma} B_\delta N^k_b + \delta_{bc} H_\delta{}^c{}_\gamma) - H_o{}^b{}_\gamma (g_{jk|\beta} y^\beta B^j{}_\delta + \delta_{bc} H_\delta{}^c{}_o).$$

§2. Totally geodesic subspaces. For the present, we assume that $M\Gamma$ is a geo-path connection, with respect to which any path in M_n is always a geodesic in M_n. In this case, it is known that the induced connection $IM\Gamma$ on M_m is also a geo-path connection on M_m and that M_m is totally geodesic if and only if each normal curvature vector vanishes i.e.,

(2. 1) $H^b_\gamma = 0 \; (b = m + 1, \cdots\cdots, n).$

Hereafter we shall restrict our connection to $T\Gamma$ (or $T\Gamma_o$) i.e., $TM\Gamma$, $TMD\Gamma$ (or $TM\Gamma_o$, $TM\Gamma_o$) for the sake of simplicity. Then we have

$$\widetilde{C}_j{}^i{}_k = C_j{}^i{}_k \text{ (or } 0), \; T^i{}_o = 0 = T^o{}_k,$$

(2. 2) $Q_o{}^i{}_k = 0 = Q_j{}^o{}_k$ (for $TM\Gamma$ (or $TM\Gamma_o$)),

$$Q_{oik} + Q_{iok} = 0 \text{ (for } TMD\Gamma \text{ (or } TMD\Gamma_o)), \; Q_{jik} = g_{is}Q_j{}^s{}_k.$$

And we know that the condition (2. 1) is equivalent to

(2. 3) $H_\beta{}^b{}_\gamma = Q_\beta{}^b{}_\gamma \; (b = m + 1, \cdots, n), \; Q_\beta{}^b{}_\gamma = Q_j{}^i{}_k B^{j\,k}_{\beta\gamma} N^b_i.$ [5]

From (2. 1) and (2. 3) we can state

Lemma 2. 1. *Let M_n be endowed with a $T\Gamma$ (or $T\Gamma_o$). Then in order that each normal curvature vector H^b_γ on M_m vanishes if and only if each second fundamental tensor $H_\beta{}^b{}_\gamma$ vanishes, it is necessary and sufficient that each tensor $Q_\beta{}^b{}_\gamma$ vanishes.*

By virtue of (1. 10) and Lemma 2. 1, we can state

Theorem 2. 1. *Let M_n be endowed with a $T\Gamma$ (or $T\Gamma_o$) and M_m be totally geodesic. Then if each tensor $Q_\beta{}^b{}_\gamma$ vanishes then the h-curvature tensor with respect to the induced connection $IT\Gamma$ (or $IT\Gamma_o$) is given as follows:*

(2. 4) $R_{\beta\alpha\gamma\delta} = R_{jikh} B^{j\,i}_{\beta\alpha} B^{k\,h}_{\gamma\delta}.$

Immediately from Theorem 2. 1 we can state

Corollary 2. 1. 1. *Let M_m be h-isotropic with R with respect to $T\Gamma$ (or $T\Gamma_o$). Then if M_m is totally geodesic and each tensor $Q_\beta{}^b{}_\gamma$ vanishes then M_m is also h-isotropic with R with respect to $IT\Gamma$ (or $IT\Gamma_o$).*

Note 2. 1. In the above Corollary, R in general is not a constant. And the range of validity of the Corollary is fairly broad.

(1) All GT-connections: $\Gamma^i{}_k = G^i{}_k + T^i{}_k, \Gamma_j{}^i{}_k = G_j{}^i{}_k + T_j{}^i{}_k.$
(or $GT(o)$-connections)

(a) If $T\Gamma = H\Gamma$ (Hashiguchi connection) then $R = 0$ or M_n is a Riemannian space of constant curvature R.

(b) If $T\Gamma_o = B\Gamma$ (Berwald connection) then M_n is of constant curvature R.

(2) All TM (or $TM(o)$)-connections with the following tensor $Q_j{}^i{}_k$:

$Q_j{}^i{}_k = u_j h^i{}_k + h^i{}_j v_k,$

where u_j and v_k are both $(o)p$-homogeneous vectors and $u_j y^j = 0$.

(a) $AMN\Gamma$ (or $AMN\Gamma_o$) : $T^i{}_k = fLh^i{}_k,\ Q_j{}^i{}_k = -Lf_{\|j}h^i{}_k$.

(b) $AMB\Gamma$ (or $AMB\Gamma_o$) : $\Gamma^i{}_k = fLh^i{}_k,\ Q_j{}^i{}_k = 2fl_k h^i{}_j - Lf_{\|j}h^i{}_k$.

where $f = f(x, y)$ is a $(o)p$-homogeneous scalar.

In particular, if f is a constant then $R = 0$ or M_n is a Riemannian space of constant curvature R [4].

(3) All TMD (or $TMD(o)$)-connections with the following tensors $Q_j{}^i{}_k$:

$Q_j{}^i{}_k = f(l_j h^i{}_k - l^i h_{jk}) + v_k h^i{}_j.$

(a) $AMBD\Gamma$ (or $AMBD\Gamma_o$) : $T^i{}_k = 0,\ Q_j{}^i{}_k = f(l_j h^i{}_k + l_k h^i{}_j - l^i h_{jk})$,

(b) $BD\Gamma$ (or $BD\Gamma_o$) : $T^i{}_k = 0,\ Q_j{}^i{}_k = f(l_j \delta^i{}_k - l^i g_{jk})$.

Contracting (2. 3) by y^β, we obtain

$$(2.5)\qquad H_o{}^b{}_\gamma = Q_o{}^b{}_\gamma\ (= D^i{}_k N_i^b B^k{}_\gamma),$$

where $D^i{}_k\ (= Q_o{}^i{}_k)$ is the deflexion tensor on M_n.

The following assumption on $IM\Gamma$ is called the D-condition :

$$(2.6)\qquad D^b{}_\gamma = 0\ (b = m+1, \cdots\cdots\cdots, n).$$

From (1. 11), (2. 1), (2. 3), (2. 5) and (2. 6), we can state

Theorem 2. 2. *Let M_n be endowed with a $T\Gamma$ (or $T\Gamma_o$) and M_m be totally geodesic. Then if the induced connection $IT\Gamma$ (or $IT\Gamma_o$) satisfied the D-condition then the contracted h-tensor $R_{o\alpha\gamma\delta}$ and $R_{o\alpha o\delta}$ with respect to $IT\Gamma$ (or $IT\Gamma_o$) are given as follows:*

$$(2.7)\qquad R_{o\alpha\gamma\delta} = R_{oikh}B^i{}_\alpha B^{kh}_{\gamma\delta},\ R_{o\alpha o\delta} = R_{oioh}B^{ih}_{\alpha\delta}.$$

Immediately from Theorem 2. 2, we can state

Corollary 2. 2. 1. *Suppose that M_n is endowed with a $T\Gamma$ (or $T\Gamma_o$). Then if M_m is totally geodesic and the induced connection $IT\Gamma$ (or $IT\Gamma_o$) satisfied the D-condition. Then if M_n is of scalar curvature R (resp. constant curvature R) with respect to $T\Gamma$ (or $T\Gamma_o$) then M_m is also of scalar curvature R (resp. constant curvature R) with respect to $IT\Gamma$ (or $IT\Gamma_o$).*

Note 2. 2. The range of validity of the above Corollary is extensive.

(1) All the *TM* (or *TM(o)*)-connections.

(2) All the *TMD* (or *TMD(o)*)-connections whose deflexion tensors are given by $D^i{}_k = fLh^i{}_k$.

§3. Totally ncd-free (or nc-constant) subspaces. In this section, we assume that M_n is endowed with a $T\Gamma$ (or $T\Gamma_o$) and the induced connection $IT\Gamma$ (or $IT\Gamma_o$) on M_m satisfies the *TDQ*-condition i.e.,

$$T^b{}_\gamma\ (= T^i{}_k B^k{}_\gamma N^b_i) = 0,\ Q_\gamma{}^b{}_o\ (= Q_j{}^i{}_k B^j{}_\gamma N^b_i y^k) = 0\ (\ b = m+1, \cdots\cdots, n\ ) \tag{3. 1}$$

together with the *D*-condition (2. 6).

Let M_m be totally ncd-free (resp. nc-constant). Then for direct-free scalars f^b ($f_{\|\gamma} = 0$) (resp. for constants f^b), we have

$$\overset{b}{H}{}^b_\gamma = L^2 f^b\ (\ b = m+1, \cdots\cdots, n\ ). \tag{3. 2}$$

And further we have the following equivalent two equations:

$$\overset{b}{H}{}^b_\gamma = f^b y_\gamma - \frac{1}{2}\bar{L}^2 \lambda^b_{c\gamma} f^c,\ y_\gamma = g_{\beta\gamma} y^\beta. \tag{3. 3}$$

$$\overset{b}{H}{}_\beta{}^b{}_\gamma = f^b g_{\beta\gamma} - f^c\,(\lambda^b_{c\beta} y_\gamma + \lambda^b_{c\gamma} y_\beta) - \frac{1}{2}\bar{L}^2 f^c\,(\lambda^b_{c\gamma\|\beta} - \lambda^b_{d\beta}\lambda^d_{c\gamma}). \tag{3. 4}$$

If we put

$$\frac{1}{2}(R_{o\gamma o\delta} + R_{o\delta o\gamma}) = \frac{1}{2}(R_{oioh} + R_{ohoi})\, B^{ih}_{\gamma\delta} + \bar{L}^2 N^2 h_{\gamma\delta} + \frac{1}{2}\Phi_{\gamma\delta}, \tag{3. 5}$$

$\Phi_{\gamma\delta} = U_{\gamma\delta} + V_{\gamma\delta}$, $N = N\,(u^\beta, y^\beta)$: normal curvature in y^β-direction, then we have [8]

$$\begin{aligned} U_{\gamma\delta} = \bar{L}^2\ \{\ &\bar{L}^2\,(f_b \lambda^b_{c\gamma}\lambda^c_{d\delta} f^d - C_{bc\delta\|\gamma} f^b f^c - \frac{1}{2}\sum_b \lambda^b_{c\gamma}\lambda^b_{d\delta} f^c f^d) \\ &- (C_{bc\gamma} y_\delta + C_{bc\delta} y_\gamma) f^b f^c\ \},\ \text{being } f_b = \delta_{bc} f^c, \end{aligned} \tag{3. 6}$$

$$\begin{aligned} V_{\gamma\delta} = -\bar{L}^2\ \{\ &T_{b\gamma\delta} + T_{b\delta\gamma} + 2(C_{\gamma b\epsilon} T^\epsilon{}_\delta + C_{\delta b\epsilon} T^\epsilon{}_\gamma + 2P_{\gamma b\delta}) \\ &+ (g_{s\gamma} B^h{}_\delta + g_{s\delta} B^h{}_\gamma) D^s{}_{h\|k} N^k_b\ \}\ f^b + \{\ \frac{1}{2}\bar{L}^2\,(T_{\gamma b}\lambda^b_{c\delta} + T_{\delta b}\lambda^b_{c\gamma}) \\ &- (T_{\gamma b} y_\delta + T_{\delta b} y_\gamma)\ \}\ f^b + g_{sj}\,(D^s{}_{o|k} - D^s{}_k)\, N^k_b\ \{\ f^b\,(B^j{}_\delta y_\gamma + B^j{}_\gamma y_\delta) \end{aligned} \tag{3. 7}$$

$$- \frac{1}{2}\bar{L}^2 (B^j{}_\gamma \lambda^b_{c\delta} + B^j{}_\delta \lambda^b_{c\gamma}) f^c \} + [-\bar{L}^2 (C_{\gamma b\epsilon} D^\epsilon{}_\delta + C_{\delta b\epsilon} D^\epsilon{}_\gamma) f^b$$

$$+ \{ C_{\epsilon\gamma b} y_\delta + C_{\epsilon\delta b} y_\gamma - \frac{1}{2}\bar{L}^2 (C_{\epsilon\gamma b}\lambda^b_{c\delta} + C_{\epsilon\delta b}\lambda^b_{c\gamma}) \} D^\epsilon{}_o f^c],$$

where all the terms in brackets vanish for $IT\Gamma_o$.

Suppose that the tensor $\Phi_{\gamma\delta}$ in (3. 5) is expressible in

(3. 8) $\quad \Phi_{\gamma\delta} = 2L^2 \mu h_{\gamma\delta}.$

Then from (3. 5) and (3. 8) we can state [8]

Theorem A. *Suppose that M_n is endowed with a $T\Gamma$ (or $T\Gamma_o$) and the induced connection $IT\Gamma$ (or $IT\Gamma_o$) satisfies the TDQ-condition and that the tensor $\Phi_{\gamma\delta}$ is expressible in (3. 8) for a scalar μ (resp. a constant μ). Then if M_n is of scalar curvature R (resp. of constant curvature R) with respect to $T\Gamma$ (or $T\Gamma_o$) and M_m is totally ncd-free (resp. nc-constant) with N then M_m is of scalar curvature $(R + N^2 + \mu)$ (resp. of constant curvature $(R + N^2 + \mu)$ with respect to $IT\Gamma$ (or $IT\Gamma_o$).*

Since the tensor $U_{\gamma\delta}$ is very complicated, we need to impose some conditions in order to simplify it. Firstly we shall consider the following assumption called the *CN*-condition [8] :

(3. 9) $\quad g_{ij} N^i_b N^j_{c\,\|\gamma} = g_{ij} N^i_c N^j_{b\,\|\gamma}$, i.e., $\lambda^b_{c\gamma} = \lambda^c_{b\gamma}$.

From (1. 5) and (3. 9), we get $\lambda^b_{c\gamma} = C_c{}^b{}_\gamma$ and hence (3. 5) becomes

(3. 10) $\quad U_{\gamma\delta} = \bar{L}^2 \{ \bar{L}^2 (\frac{1}{2} C_{bd\gamma} C_c{}^d{}_\delta - C_{bc\delta\|\gamma}) - (C_{bc\gamma} y_\delta + C_{bc\delta} y_\gamma) \} f^b f^c,$

where $C_{bc\gamma} = C_{ijk} N^i_b N^j_c B^k{}_\gamma = C_b{}^c{}_\gamma = C_c{}^b{}_\gamma$.

Secondary we consider another assumption called the *SN*-condition

(3. 11) $\quad C_{bc\gamma} f^b = 0$ i.e., $g_{ij} N^i_b \bar{N}^j{}_{\|\gamma} = 0$, being $\bar{N}^j = N^j_b f^b$.

Then it follows from (3. 10) and (3. 11) that the tensor $U_{\gamma\delta}$ vanishes under both the *CN*- and *SN*-conditions.

Now we shall consider a special *TM* (or *TM(o)*)-connection $TMA\Gamma$ (or $TMA\Gamma_o$), whose non-linear connection is defined by $\Gamma^i{}_k = G^i{}_k + fLh^i{}_k$.

In this case, we have

$$T^i{}_k = fLh^i{}_k,\ T_j{}^i{}_k = Lf_{\|j} h^i{}_k + f(l_j h^i{}_k - l_k h^i{}_j - l^i h_{jk}),$$

(3. 12) $\quad T_{jk} = g_{is} T^s{}_k,\ T_{jik} = g_{is} T_j{}^s{}_k,\ Q_o{}^i{}_k = D^i{}_k = 0 = Q_j{}^o{}_k,$

$$T^{b}{}_{\gamma} = 0,\ T_{\beta}{}^{b}{}_{\gamma} = 0,\ T^{\varepsilon}{}_{\gamma} = f\bar{L}h^{\varepsilon}{}_{\gamma},\ T_{b\gamma\delta} = \bar{L}f_{\|j}N^{j}_{b}h_{\gamma\delta}.$$

Applying (3. 12) to (3. 7), we obtain

$$V_{\gamma\delta} = -\bar{L}^{2}\{\,2f_{\|j}N^{j}_{b}h_{\gamma\delta} + 4\,(C_{\gamma b\delta}f\bar{L} + P_{\gamma b\delta})\,\}f^{b}. \tag{3. 13}$$

If we apply (3. 10) and (3. 13) to (3. 8) through (3. 5) then we get

$$\begin{aligned}(f_{\|j}N^{j}_{b}f^{b} + \mu)h_{\gamma\delta} + 2(C_{\gamma b\delta}f\bar{L} + P_{\gamma b\delta})f^{b} + \frac{1}{2}\bar{L}^{2}\,(C_{bc\gamma}y_{\delta} + \\ + C_{bc\delta}y_{\gamma} + C_{bc\delta\|\gamma} - \frac{1}{2}C_{bd\gamma}C^{d}_{c}{}_{\delta})f^{b}f^{c} = 0.\end{aligned} \tag{3. 14}$$

Under both the CN- and SN-conditions, the above (3. 14) reduces to

$$(f_{\|j}N^{j}_{b}f^{b} + \mu)\,h_{\gamma\delta} + 2(C_{\gamma b\delta}f\bar{L} + P_{\gamma b\delta})f^{b} = 0. \tag{3. 15}$$

Thus from (3. 14), (3. 15) and Theorem A, we can state

Theorem 3. 1. *Suppose that M_n is endowed with a $TMA\Gamma$ (or $TMA\Gamma_o$) and the induced connection $ITMA\Gamma$ (or $ITMA\Gamma_o$) satisfies the Q-condition ($Q_{\beta}{}^{b}{}_{o} = 0$). Then if M_n is of scalar curvature R (resp. of constant curvature R) with respect to $TMA\Gamma$ (or $TMA\Gamma_o$) and M_m is totally ncd-free (resp. nc-constant) with N then M_m is of scalare curvature ($R + N^2 + \mu$) (resp. of constant curvature ($R + N^2 + \mu$)) with respect to $ITMA\Gamma$ (or $ITMA\Gamma_o$) under the following facts :*

(1) *For a scalar μ (resp. a constant μ), the relation* (3. 14) *holds under the CN-condition.*

(2) *For a scalar μ (resp. a constant μ), the relation* (3. 15) *holds under both CN- and SN-conditions.*

Note 3. 1. In the above Theorem, the tensor $Q_{j}{}^{i}{}_{k}$ may be written in

$$Q_{j}{}^{i}{}_{k} = W_{j}{}^{i}{}_{k} + h^{i}{}_{j}v_{k},$$

where $W_{j}{}^{i}{}_{k}$ is a $(o)p$-homogeneous tensor satisfying $W_{o}{}^{i}{}_{k} = W_{j}{}^{i}{}_{o} = W_{j}{}^{o}{}_{k} = 0$ and v_k is a $(o)p$-homogeneous vector. Examples are as follows :

(1) $AMN\Gamma$ (or $AMN\Gamma_o$) : $Q_{j}{}^{i}{}_{k} = -Lf_{\|j}h^{i}{}_{k}\ (v_k = 0)$.

(2) $AMB\Gamma$ (or $AMB\Gamma_o$) : $Q_{j}{}^{i}{}_{k} = -2fl_{k}h^{i}{}_{j} - Lf_{\|j}h^{i}{}_{k}\ (v_k = -2fl_k)$.

(3) $AMC\Gamma$ (or $AMC\Gamma_o$) : $Q_{j}{}^{i}{}_{k} = 2fl_{k}h^{i}{}_{j} - Lf_{\|j}h^{i}{}_{k} - P_{j}{}^{i}{}_{k}\ (v_k = 2fl_k)$.

(4) $AMR\Gamma$ (or $AMR\Gamma_o$) : $Q_{j}{}^{i}{}_{k} = -Lf_{\|j}h^{i}{}_{k} + fl_{k}h^{i}{}_{j} - fLC_{j}{}^{i}{}_{k} - P_{j}{}^{i}{}_{k}\ (v_k = fl_k)$.

We shall call a $TMD\Gamma$ (or $TMD\Gamma_o$) a $AM\tilde{D}\Gamma$ (or $AM\tilde{D}\Gamma_o$) if the tensors $T^i{}_k$ and $D^i{}_k$ are given by $T^i{}_k = fLh^i{}_k$ and $D^i{}_k = -T^i{}_k$. Then the h-connection is expressible in $\Gamma_j{}^i{}_k = G_j{}^i{}_k$, where $W_j{}^i{}_k$ is a (o) p-homogeneous tensor satisfying $W_o{}^i{}_k = 0 = W_j{}^o{}_k$. And we have

$$\Gamma^i{}_k = G^i{}_k + T^i{}_k,\ Q_j{}^i{}_k = W_j{}^i{}_k - T_j{}^i{}_k,\ Q_o{}^i{}_k = -T^i{}_k,$$

$$Q_j{}^o{}_k = T_{jk},\ Q_j{}^i{}_o = W_j{}^i{}_o + T^i{}_j,\ T^b{}_\gamma = 0, \tag{3. 16}$$

$$g_{s\gamma}B^h{}_\delta D^s{}_{h\|k}N^n_b f^b = -g_{s\gamma}B^h{}_\delta T_k{}^s{}_h N^k_b f^b = -T_{b\gamma\delta}.$$

Applying (3. 16) to (3. 7), we obtain

(3. 17)
$$V_{\gamma\delta} = -2\bar{L}^2(f\bar{L}C_{\gamma b\delta} + 2P_{\gamma b\delta})f^b \qquad (\text{for } AM\tilde{D}\Gamma),$$
$$V_{\gamma\delta} = -4\bar{L}^2(f\bar{L}C_{\gamma b\delta} + P_{\gamma b\delta})f^b \qquad (\text{for } AM\tilde{D}\Gamma_o).$$

From (3. 17), we have in the same way as before

(3. 18)
$$2\{\mu h_{\gamma\delta} + (f\bar{L}C_{\gamma b\delta} + 2P_{\gamma b\delta})f^b\} + \Omega_{\gamma\delta} = 0 \qquad (\text{for } AM\tilde{D}\Gamma),$$
$$2\{\mu h_{\gamma\delta} + 2(f\bar{L}C_{\gamma b\delta} + P_{\gamma b\delta})f^b\} + \Omega_{\gamma\delta} = 0 \qquad (\text{for } AM\tilde{D}\Gamma_o),$$
$$\Omega_{\gamma\delta} = \{\bar{L}^2(\tfrac{1}{2}C_{bd\gamma}C_c{}^d{}_\delta - C_{bc\delta\|\gamma}) - (C_{bc\gamma}y_\delta + C_{bc\delta}y_\gamma)\}f^b f^c,$$

(3. 19)
$$\mu h_{\gamma\delta} + (f\bar{L}C_{\gamma b\delta} + 2P_{\gamma b\delta})f^b = 0 \qquad (\text{for } AM\tilde{D}\Gamma),$$
$$\mu h_{\gamma\delta} + 2(fLC_{\gamma b\delta} + P_{\gamma b\delta})f^b = 0 \qquad (\text{for } AM\tilde{D}\Gamma_o).$$

From (3. 18), (3. 19) and Theorem A, we can state

Theorem 3. 2. *Suppose that M_n is endowed with an $AM\tilde{D}\Gamma$ (or $AM\tilde{D}\Gamma_o$) and the induced connection $IAM\tilde{D}\Gamma$ (or $IAM\tilde{D}\Gamma_o$) satisfies the Q-condition. Then if M_n is of scalar curvature R (resp. of constant curvature R) with respect to $AM\tilde{D}\Gamma$ (or $AM\tilde{D}\Gamma_o$) and M_m is totally ncd-free (resp. nc-constant) with N then M_m is of scalar curvature $(R + N^2 + \mu)$ (resp. of constant $(R + N^2 + \mu)$) with respect to $IAM\tilde{D}\Gamma$ (or $IAM\tilde{D}\Gamma_o$) under the following facts :*

(1) *For a scaler μ (resp. a constant μ), the relation* (3. 18) *holds under the CN-condition.*

(2) *For a scalar μ (resp. a constant μ), the relation* (3. 19) *holds under both the CN- and SN-conditions.*

Note 3. 2. In the above Theorem, the tensor $W_j{}^i{}_k$ may be written in

$$W_j{}^i{}_k = \overline{W}_j{}^i{}_k + h^i{}_j v_k,$$

Where $\overline{W}_j{}^i{}_k$ is a $(o)p$-homogeneous tensor satisfying $\overline{W}_o{}^i{}_k = \overline{W}_j{}^i{}_o = \overline{W}_j{}^o{}_k = 0$ and v_k is a $(o)p$-homogeneous vector. An example is as follows :

$$AMD\Gamma \text{ (or } AMD\Gamma_o) : \overline{W}_j{}^i{}_k = -fLC_j{}^i{}_k - P_j{}^i{}_k,\ v_k = 0.$$

We shall call a $TMD\Gamma$ (or $TMD\Gamma_o$) a $GQDA\Gamma$ (or $GQDA\Gamma_o$) if its non-linear and h-connections $\Gamma^i{}_k$ and $\Gamma_j{}^i{}_k$ are defined by

$$\text{(3. 20)} \quad \Gamma^i{}_k = G^i{}_k,\ \Gamma_j{}^i{}_k = G_j{}^i{}_k + Q_j{}^i{}_k,\ Q_o{}^i{}_k = D^i{}_k = fLh^i{}_k,\ Q_j{}^o{}_k = -g_{js}D^i{}_k.$$

Then it follows from (3. 20) that

$$\text{(3. 21)} \quad T^i{}_k = 0,\ g_{s\gamma}B^h{}_\delta D^s{}_{h\|k}N^k_b f^b = Lf_{\|k}N^k_b f^b.$$

From (3. 7), (3. 20) and (3. 21) we obtain

$$\text{(3. 22)} \quad \begin{aligned} V_{\gamma\delta} &= -2\bar{L}^2(\bar{L}f_{\|k}N^k_b h_{\gamma\delta} + \bar{L}fC_{\gamma b\delta} + 2P_{\gamma b\delta})f^b && \text{(for } GQDA\Gamma), \\ V_{\gamma\delta} &= -2\bar{L}^2(Lf_{\|k}N^k_b h_{\gamma\delta} + 2P_{\gamma b\delta})f^b && \text{(for } GQDA\Gamma_o). \end{aligned}$$

From (3. 22), we get in the same way as before

$$\text{(3. 23)} \quad \begin{aligned} &2(\bar{L}f_{\|k}N^k_b f^b + \mu)h_{\gamma\delta} + 2(f\bar{L}C_{\gamma b\delta} + 2P_{\gamma b\delta})f^b + \Omega_{\gamma\delta} = 0 && \text{(for } GQDA\Gamma), \\ &2(\bar{L}f_{\|k}N^k_b f^b + \mu)h_{\gamma\delta} + 4P_{\gamma b\delta}f^b + \Omega_{\gamma\delta} = 0 && \text{(for } GQDA\Gamma_o), \end{aligned}$$

$$\text{(3. 24)} \quad \begin{aligned} &(\bar{L}f_{\|k}N^k_b f^b + \mu)h_{\gamma\delta} + 2(f\bar{L}C_{\gamma b\delta} + 2P_{\gamma b\delta})f^b = 0 && \text{(for } GQDA\Gamma), \\ &(\bar{L}f_{\|k}N^k_b f^b + \mu)h_{\gamma\delta} + 2P_{\gamma b\delta}f^b = 0 && \text{(for } GQDA\Gamma_o). \end{aligned}$$

Hence from (3. 23), (3. 24) and Theorem A, we can state

Theorem 3. 3. *Suppose that M_n is endowed with a $GQDA\Gamma$ (or $GQDA\Gamma_o$) and the induced connection $IGQDA\Gamma$ (or $IGQDA\Gamma_o$) satisfies the Q-condition. Then if M_n is of scalar curvature R (resp. of constant curvature R) with respect to $GQDA\Gamma$ (or $GQDA\Gamma_o$) and M_m is totally ncd-free (resp. nc-constant) with N then M_m is of scalar curvature $(R + N^2 + \mu)$ (resp. of constant curvature $(R + N^2 + \mu)$) with respect to $IGQDA\Gamma$ (or $IGQDA\Gamma_o$) under the following facts :*

(1) *For a scalar μ (resp. a constant μ), the relation* (3. 23) *holds under CN-condition.*

(2) *For a scalar* μ *(resp. a constant* μ*), the relation* (3. 24) *holds under both the CN- and SN-conditions.*

Note 3. 3. In the above Theorem, the tensor $Q_j{}^i{}_k$ may be written in

$$Q_j{}^i{}_k = f(l_j h^i{}_k - l^i h_{jk}) + h^i{}_j v_k + W_j{}^i{}_k,$$

where v_k is a $(1)p$-homogeneous vector and $W_j{}^i{}_k$ is a $(o)p$-homogeneous tensor satisfying $W_o{}^i{}_k = W_j{}^i{}_o = W_j{}^o{}_k = 0$. Examples are as follows :

(1) $MD\Gamma$ (or $MD\Gamma_o$) : $Q_j{}^i{}_k = f(l_j h^i{}_k - l^i h_{jk}) - P_j{}^i{}_k (v_k = 0,\ W_j{}^i{}_k = -P_j{}^i{}_k)$

(2) $AMBD\Gamma$ (or $AMBD\Gamma_o$) : $Q_j{}^i{}_k = f(l_j h^i{}_k - l^i h_{jk} + l_k h^i{}_j)\ (v_k = l_k,\ W_j{}^i{}_k = 0)$

(3) $AMCD\Gamma$ (or $AMCD\Gamma_o$) : $Q_j{}^i{}_k = f(l_j h^i{}_k - l^i h_{jk} + l_k h^i{}_j) - P_j{}^i{}_k$
$(v_k = l_k,\ W_j{}^i{}_k = -P_j{}^i{}_k)$.

REFERENCES

[1] M. Matsumoto : The induced and intrinsic Finsler connections of hypersurfaces and Finslerien projective geometry, *J. Math. Kyoto Univ.,* **25** (1985), 107-144.

[2] M. Matsumoto : Foundations of Finsler geometry and special Finsler spaces, (1986), *Kaiseisha press, Japan.*

[3] H. Yasuda : On TM-connections of a Finsler space and the induced connections of its hypersurfaces, *Proc. Rom. – Jap. Col. Finsler Geo., Braçov* (1984), 183-232.

[4] H. Yasuda and M. Yoshida : On special hypersurfaces of a Finsler spaces, *Tensor, N.S.,* **42** (1985), 283-298.

[5] H. Yasuda : A theory of subspaces in a Finsler space, *Ann, Rep. Asahikawa Med. Coll.,* **7** (1987), 1-43.

[6] H. Yasuda : Special subspaces in a Finsler space, *Ann, Rep. Asahikawa Med. Coll.,* **9** (1988), 1-14.

[7] H. Yasuda : Connections on subspaces in a Finsler space, *Ann. Rep. Asahikawa Med. Coll.,* **10** (1989), 1-12.

[8] H. Yasuda and T. Yamada : A study of special subspaces in a Finsler space, *Ann. Rep. Asahikawa Med. Coll.,* **11** (1990), 1-12.

(Mathematics, Asahikawa Medical College)

(Mathematics, Banaras Hindu University, Varananasi (U.P.), India)