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CONNECTIONS ON SUBSPACES IN A FINSLER SPACE

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Introduction. In the previous paper ([3], [4])[1], we have developed the theories of subspaces in a Finsler space M through three kinds of connections ($M\Gamma$, $T\text{-}MD\Gamma$ and $TM\Gamma$) and their induced connections ($IM\Gamma$, $ITMD\Gamma$, $ITM\Gamma$). For investigating subspaces, two connections are usually considered. One is the induced connection and the other is the intrinsic one (if it can be defined). In general, the properties of the former become worse than those of the original connection on M. On the other hand, the theory constructed through the latter is more complicated. Therefore we have introduced some intermediate connections (called semi-induced connections) in [3] and [5].

The principal purpose of the present paper is to study the above connections. The terminologies and notations refer to papers [2] ～ [5].

§1. Preliminaries. Let M_n be an n-dimensional Finsler space with a fundamental function $L(x^i, y^i)$, and be endowed with a Matsumoto connection $M\Gamma = (\Gamma_j{}^i{}_k, \Gamma^i{}_k, \widetilde{C}_j{}^i{}_k)$, where $\widetilde{C}_j{}^i{}_k$ is a (-1) p-homogeneous[2] tensor. The non-linear connection $\Gamma^i{}_k$ and the h-connection $\Gamma_j{}^i{}_k$ are given by

$$(1.1) \qquad \Gamma^i{}_k = G^i{}_k + T^i{}_k,\ \ \Gamma_j{}^i{}_k = \Gamma^i{}_{k\|j} + Q_j{}^i{}_k = G_j{}^i{}_k + T_j{}^i{}_k + Q_j{}^i{}_k,$$

where the symbol $\|j$ indicates the partial differentiation by y^j, $G^i{}_k$ and $G_j{}^i{}_k$ $(= G^i{}_{k\|j})$ are the non-linear connection and the h-connection of Berwald, $T^i{}_k$ and $Q_j{}^i{}_k$ are $(1)p$- and $(0)p$-homogeneous tensors respectively and $T_j{}^i{}_k = T^i{}_{k\|j}$. Now we

1) Numbers in brackets refer to the references at the end of the paper.
2) "$(r)p$-homogeneous" means "positively homogeneous of degree r in y^i".

consider an m-dimensional subspace M_m of M_n, which is represented by the equation

$$(1.2)\qquad x^i = x^i(u^\alpha)\ (i=1,\cdots\cdots,\ n;\ \alpha=1,\cdots\cdots,m),$$

where we suppose that variables u^α form a coordinate system of M_m and the matrix with components $B^i_\alpha(=\partial x^i/\partial u^\alpha)$ is of rank m.

Let us denote the components of a vector y^i tangent to a curve in M_m by y^α [3] in terms of u^α-system. Then we have

$$(1.3)\qquad y^i = B^i_\alpha y^\alpha,\ y^i{}_{\|\alpha} := \partial y^i/\partial y^\alpha = B^i_\alpha.$$

Further the induced fundamental function $\bar{L}(u^\alpha, y^\alpha)$ and the metric tensor $g_{\beta\gamma}(u^\alpha, y^\alpha)$ on M_m are given by

$$(1.4)\qquad \bar{L} = L(x^i(u^\alpha), B^i_\alpha y^\alpha),\ g_{\beta\gamma} = g_{jk}B^{jk}_{\beta\gamma} := g_{jk}B^j_\beta B^k_\gamma,$$

where g_{jk} is the metric tensor on M_n.

Now we shall choose n-m unit normal vectors N^i_a $(a = m+1,\cdots\cdots,n)$ at each point (u^α) of M_m as follows:

$$(1.5)\qquad g_{ij}N^i_a N^j_b = \delta_{ab}.\quad B^i_\alpha N^b_i = O,\ N^b_i := g_{ij}N^j_b.$$

Put $B^\beta_i = g_{ij}B^j_\gamma g^{\gamma\beta}$, where $g^{\gamma\beta}$ is the reciprocal tensor of $g_{\gamma\beta}$. Then (B^β_i, N^b_i) is the inverse matrix of (B^i_β, N^i_b). In this case, we have the following relations:

$$(1.6)\qquad B^\beta_{i\|\gamma} = C^\beta_{b\gamma}N^b_i,\ N^i_{b\|\gamma} = -2C^\beta_{b\gamma}B^i_\beta - \lambda^c_{b\gamma}N^i_c,\ N^b_{i\|\gamma} = \lambda^b_{c\gamma}N^c_i,$$

$$(1.7)\qquad \lambda^c_{b\gamma} + \lambda^b_{c\gamma} = 2C^c_{b\gamma} = 2C^b_{c\gamma},\ \lambda^c_{b\gamma} := -N^c_i N^i_{b\|\gamma},$$

where $C^\beta_{b\gamma} = C^i_{jk}N^j_b B^\beta_i B^k_\gamma$, $C^c_{b\gamma} = C^i_{jk}N^c_i N^j_b B^k_\gamma$ and $C^s_{jk}g_{si} = C_{jik}\,(=\frac{1}{2}g_{ji\|k})$.

The induced Matsumoto connection $IM\Gamma = (\Gamma^\alpha_{\beta\gamma}, \Gamma^\alpha_\gamma, \widetilde{C}^\alpha_{\beta\gamma})$ on M_m is defined as follows:

$$(1.8)\qquad \Gamma^\alpha_\gamma = B^\alpha_i(B^i_{0\gamma} + \Gamma^i_k B^k_\gamma),\ \widetilde{C}^\alpha_{\beta\gamma} = \widetilde{C}^i_{jk}B^\alpha_i B^{jk}_{\beta\gamma},$$

$$(1.9)\qquad \Gamma^\alpha_{\beta\gamma} = B^\alpha_i(B^i_{\beta\gamma} + \Gamma^i_{jk}B^{jk}_{\beta\gamma}) + \widetilde{C}^\alpha_{\beta b}H^b_\gamma,$$

where

3) If no confusion occurs, then we shall use y^α in stead of the usual notation v^α

(1.10) $B^i_{\beta\gamma} = \partial B^i{}_\gamma / \partial u^\beta$, $B^i_{o\gamma} = B^i_{\beta\gamma} y^\beta$, $\widetilde{C}^a_{\beta b} = \widetilde{C}_j{}^i{}_k B_i^a B^k{}_\beta N^k{}_b$,

(1.11) $H^a{}_\gamma = N^a{}_i (B^i_{o\gamma} + \Gamma^i{}_k B^k{}_\gamma)$.

The normal curvature vector in a normal direction $N^i{}_a$ is given by (1.11), while the second fundamental tensor in the same direction is given by

(1.12) $H^a_{\beta\gamma} = N^a{}_i (B^i_{\beta\gamma} + \Gamma_j{}^i{}_k B^{jk}_{\beta\gamma}) + \widetilde{C}^a_{\beta b} H^b{}_\gamma$, $\widetilde{C}^a_{\beta b} := \widetilde{C}_j{}^i{}_k B^j{}_\beta N^k{}_b N^a{}_i$.

Let $R_{\alpha\delta\beta\gamma}$ be the h-curvature tensor with respect to $IM\Gamma$, which is calculated in the paper [3] by means of the generalized Gauss equation. If we contract this tensor by $y^\alpha y^\beta$, we have

$$
\begin{aligned}
R_{o\delta o\gamma} &= R_{oioh} B^{ih}_{\delta\gamma} + S_{oikh} B^i{}_\delta N^k{}_a N^h{}_b H^a{}_o H^b{}_\gamma + B^i{}_\delta N^h{}_a (P_{oioh} H^a{}_\gamma - P_{oikh} B^k{}_\gamma H^a{}_o) \\
&+ H^a_{oo} (g_{jk|\gamma} B^j{}_\delta N^k{}_a + \delta_{ab} H^b_{\delta\gamma}) - H^a_{o\gamma} (g_{jk|\beta} y^\beta B^j{}_\delta N^k{}_a + \delta_{ab} H^b_{\delta o}),
\end{aligned}
\tag{1.13}
$$

where $R_{oioh} = R_{jikh} y^j y^k$, $S_{oikh} = S_{jikh} y^j$, $P_{oioh} = P_{jikh} y^j y^k$, $H^b{}_o = H^b{}_\gamma y^\gamma$, $H^a_{oo} = H^a_{\beta\gamma} y^\beta y^\gamma$, $H^a_{o\gamma} = H^a_{\beta\gamma} y^\beta$, $H^a_{\delta o} = H^a_{\delta\gamma} y^\gamma$ and R_{jikh}, S_{jikh}, P_{jikh} are the h-, v-, hv-curvature tensors with respect to $M\Gamma$ respectively.

§ 2. Semi-induced Matsumoto connections.

We take a suitable $(O)p$-homogeneous tensor $E^a_{\beta\gamma}$ on M_m and put

(2.1) $\widetilde{\Gamma}^a_{\beta\gamma} = \Gamma^a_{\beta\gamma} + E^a_{\beta\gamma}$, $\widetilde{IM\Gamma} = (\widetilde{\Gamma}^a_{\beta\gamma}, \Gamma^a{}_\gamma, \widetilde{C}^a_{\beta\gamma})$,

where $\Gamma^a_{\beta\gamma}$, $\Gamma^a{}_\gamma$ and $\widetilde{C}^a_{\beta\gamma}$ are the h-, non-linear and v-connections of $IM\Gamma$ respectively.

We shall call the above connection $\widetilde{IM\Gamma}$ a *semi-induced Matsumoto connection*. We denote the h-covariant differentiations with respect to $\widetilde{IM\Gamma}$ and $IM\Gamma$ by the symbols $;$ and $|$ respectively. As for the v-covariant differentions with respect to the above two connections, they are the same and denoted by the symbol $\|$. For the vectors related by (1.3) and $X^i = B^i{}_a X^a$, the following relations hold :

(2.2) $Dy^a = B_i^a Dy^i$, $DX^a = B_i^a DX^i$,

(2.3) $\widetilde{D}X^a = X^a{}_{;\gamma} du^\gamma + X^a \|_\gamma Dy^\gamma = DX^a + E^a_{\beta\gamma} X^\beta du^\gamma$,

where

(2.4) $$Dy^i = dy^i + \Gamma^i{}_k dx^k, \quad DX^i = X^i|_k dx^k + X^i|_k Dy^k,$$

(2.5) $$Dy^a = dy^a + \Gamma^a{}_\gamma du^\gamma, \quad DX^a = X^a{}_{|\gamma} du^\gamma + X^a|_\gamma Dy^\gamma.$$

Let $X^{ia}_{j\beta}$ be an object defined on M_m such that it is a tensor on M_n of type (1, 1) and, at the same time, a tensor on M_m of type (1, 1). Then with respect to $\widetilde{IM\Gamma}$, the relative h-covariant derivative of the above object is defined as follows:

(2.6)
$$X^{ia}_{j\beta\,;\gamma} = \delta_\gamma X^{ia}_{j\beta} + X \quad \Gamma^i_{k\gamma} - X^{ia}_{k\beta}\Gamma^k_{j\gamma} + X^{i\varepsilon}_{j\beta}\widetilde{\Gamma}^a_{\varepsilon\gamma} - X^{ia}_{j\varepsilon}\widetilde{\Gamma}^\varepsilon_{\beta\gamma}$$
$$= X^{ia}_{j\beta|\gamma} + X^{i\varepsilon}_{j\beta}E^a_{\varepsilon\gamma} - X^{ia}_{j\varepsilon}E^\varepsilon_{\beta\gamma},$$

where δ_γ and $\Gamma^i_{k\gamma}$ are the same as in $IM\Gamma$ and $X^{ia}_{j\beta|\gamma}$ is the relative h-covariant one with respect to $IM\Gamma$.

The h-curvature tensor $\widetilde{R}_{\beta a\gamma\delta}(= g_{\varepsilon a}\widetilde{R}^\varepsilon_{\beta\gamma\delta})$ of $\widetilde{IM\Gamma}$ is given by

(2.7) $$\widetilde{R}_{\beta a\gamma\delta} = R_{\beta a\gamma\delta} + E_{\beta a\varepsilon}\tau_\gamma{}^\varepsilon{}_\delta + [g_{a\varepsilon}E^\varepsilon_{\beta\gamma|\delta} + E^\varepsilon_{\beta\gamma}E_{\varepsilon a\delta} - \gamma\,|\,\delta\,],$$

where $E_{\beta a\varepsilon} = g_{a\gamma}E^\gamma_{\beta\varepsilon}$, $\tau_\gamma{}^\varepsilon{}_\delta = \Gamma^\varepsilon_{\gamma\delta} - \Gamma^\varepsilon_{\delta\gamma}$ and the symbol $\gamma\,|\,\delta$ means the interchange of indices γ and δ in the foregoing terms.

Contracting (2.7) by $y^\beta y^\gamma$, we have

(2.8)
$$\widetilde{R}_{oa o\delta} = R_{oao\delta} + E_{oa\varepsilon}\tau_o{}^\varepsilon{}_\delta + g_{a\varepsilon}(E^\varepsilon_{o\,o|\delta} - E^\varepsilon_{\beta o}y^\beta{}_{|\delta} - E^\varepsilon_{o\gamma}y^\gamma{}_{|\delta} - E^\varepsilon_{o\delta|\gamma}y^\gamma$$
$$+ E^\varepsilon_{\beta\delta}y^\beta{}_{|\gamma}y^\gamma) + E^\varepsilon_{oo}E_{\varepsilon a\delta} - E^\varepsilon_{o\delta}E_{\varepsilon ao},$$

where the index o means the contraction by the vector y, for example $R_{oao\delta} = R_{\beta a\gamma\delta}y^\beta y^\gamma$ and $E_{oa\varepsilon} = E_{\beta a\varepsilon}y^\beta$ etc.

From (1.1) and (1.8), we have

(2.9) $$y^\beta{}_{|\gamma} = D^\beta{}_\gamma + \widetilde{C}^\beta_{o\,b}H^b{}_\gamma, \quad y^\beta{}_{|\gamma}y^\gamma = D^\beta{}_o + \widetilde{C}^\beta_{o\,b}H^b{}_o,$$

where $D^\beta{}_\gamma = D^i{}_k B^{\beta k}_{i\gamma}$ and $D^i{}_k(= Q_j{}^i{}_k y^j)$ is the deflexion tensor of $M\Gamma$.

Applying (2.6) to $B^i{}_\beta$ and using (1.12), we obtain

(2.10) $$B^i{}_{\beta;\gamma} = B^i{}_{\beta|\gamma} - B^i{}_\varepsilon E^\varepsilon_{\beta\gamma} = N^i{}_b H^b_{\beta\gamma} - B^i{}_\varepsilon E^\varepsilon_{\beta\gamma}.$$

In view of (2.10), the tensor $H^b_{\beta\gamma}$ may be considered as the second fundamental tensor in a direction $N^i{}_b$ with respect to $\widetilde{IM\Gamma}$.

In [3], we have considered a totally auto-parallel subspace with respect to

$IM\Gamma$. Also with respect to $\widetilde{IM\Gamma}$, we can consider the same subspace, in which each path in M_m with respect to $\widetilde{IM\Gamma}$ is always a path in M_n with respect to $M\Gamma$. In this case, Theorem 7.1 in [3] holds also with respect to $\widetilde{IM\Gamma}$.

An $M\Gamma$ is called a geo-path connection if any path with respect to the $M\Gamma$ is always a geodesic in M_n. If M_n is endowed with a geo-path connection $M\Gamma$, then it is seen that $\widetilde{IM\Gamma}$ is also a geo-path connection on M_m and any totally auto-parallel subspace becomes a totally geodesic subspace. In this case, Theorem 7.2 in [3] of course holds.

§3. Semi-induced STD-connection. Let M_n be endowed with an $STD\Gamma$ (*or* $STD\Gamma_0$), which are defined as follows [3]:

(3.1) $$\Gamma^i_k = G^i_k + T^i_k, \quad \widetilde{C}_j{}^i{}_k = C_j{}^i{}_k (\text{or } \widetilde{C}_j{}^i{}_k = 0),$$

(3.2) $$\Gamma_j{}^i{}_k = \Gamma^*{}_j{}^i{}_k + W_j{}^i{}_k, \quad W_j{}^i{}_k = C_{jkr}T^{ri} - C_j{}^i{}_r T^r{}_k - C_k{}^i{}_r T^r{}_j$$

where T^i_k is a $(1)p$-homogeneous tensor satisfying $T^o_k = T^i_o = 0$, $\Gamma^*{}_j{}^i{}_k$ is the h-connection of Cartan and $T^{ri} = T^r{}_s g^{si}$.

These connections have the following properties:

(3.3)

(1) metrical ($L_{|k} = 0$) (2) h-metrical ($g_{ij|k} = 0$)

(3) geo-path connection (4) h-symmetric ($\Gamma_j{}^i{}_k = \Gamma_k{}^i{}_j$)

(5) v-metrical ($g_{ij}|_k = 0$) (for $STD\Gamma$)

(5)′ $g_{ij}|_k = g_{ij\|k} = 2C_{ijk}$ (for $STD\Gamma_0$)

We put

(3.4) $$T^a{}_\gamma = T^i{}_k B^{ak}_{i\gamma}, \quad C_\gamma{}^a{}_b = C_k{}^i{}_j N^j{}_b B^{ka}_{\gamma i} (= C_b{}^a{}_\gamma), \quad 2G^i = G^i{}_k y^k,$$
$$\overset{b}{H}{}^b{}_0 = N^b{}_i (B^i_{00} + 2G^i), \quad \widetilde{T}^a{}_\gamma = T^a{}_\gamma - C_\gamma{}^a{}_b H^b{}_0, \quad C_\beta{}^a{}_\gamma = C_j{}^i{}_k B^a_i B^{jk}_{\beta\gamma}.$$

In this cace, the induced connections $ISTD\Gamma = (\Gamma_\beta{}^a{}_\gamma, \Gamma^a{}_\gamma, C_\beta{}^a{}_\gamma)$ and $ISTD\Gamma_0 = (\overset{0}{\Gamma}{}_\beta{}^a{}_\gamma, \Gamma^a{}_\gamma, 0)$ are given by

(3.5) $$\Gamma^a{}_\gamma = G^a{}_\gamma + \widetilde{T}^a{}_\gamma \quad \text{(in common)},$$
$$\Gamma_\beta{}^a{}_\gamma = \overset{0}{\Gamma}{}_\beta{}^a{}_\gamma + C_\beta{}^a{}_b H^b{}_\gamma, \quad \overset{0}{\Gamma}{}_\beta{}^a{}_\gamma = B^a_i (B^i_{\beta\gamma} + \Gamma_j{}^i{}_k B^{jk}_{\beta\gamma})$$

where $G^a{}_\gamma$ is the intrinsic non-linear connection of Cartan on M_m.

Since $\widetilde{T}^0{}_\gamma = \widetilde{T}^a{}_0 = 0$, by means of this tensor $\widetilde{T}^a{}_\gamma$ we can define an $STD\Gamma$

(or $STD\Gamma_o$) on M_m, which is called the semi-induced STD(or $STD(O)$)-connection and denoted by $\widetilde{I}STD\Gamma$(or $\widetilde{I}STD\Gamma_o$). In this case, we have

$$
\begin{aligned}
\widetilde{\Gamma}^{a}_{\beta\gamma} &= \Gamma^{*a}_{\beta\gamma} + C_{\beta\gamma\varepsilon}\widetilde{T}^{\varepsilon a} - C^{a}_{\beta\varepsilon}\widetilde{T}^{\varepsilon}_{\gamma} - C^{a}_{\gamma\varepsilon}\widetilde{T}^{\varepsilon}_{\beta} \\
&= \Gamma^{a}_{\beta\gamma} + E^{a}_{\beta\gamma} = \overset{o}{\Gamma}{}^{a}_{\beta\gamma} + \overset{o}{E}{}^{a}_{\beta\gamma},
\end{aligned} \tag{3.6} \quad [3]
$$

where

$$
E^{a}_{\beta\gamma} = C^{a}_{\gamma b}H^{b}_{\beta} - C_{\beta\gamma b}H^{ba}, \quad \overset{o}{E}{}^{a}_{\beta\gamma} = E^{a}_{\beta\gamma} + C^{a}_{\beta b}H^{b}_{\gamma}, \tag{3.7}
$$

$\widetilde{T}^{\varepsilon a} = \widetilde{T}^{\varepsilon}_{\delta}g^{\delta a}$, $H^{ba} = H^{b}_{\varepsilon}g^{\varepsilon a}$ and $\Gamma^{*a}_{\beta\gamma}$ is the intrinsic h-connection of Cartan on M_m.

For $ISTD\Gamma$ and $ISTD\Gamma_o$, from (2.9), (3.1) ~ (3.5) and (3.7) we have

$$
\begin{aligned}
& y^{a}{}_{|\gamma} = -T^{a}{}_{\gamma}, \quad y^{a}{}_{|\gamma}y^{\gamma} = 0 \quad \text{(in common)}, \\
& g_{\beta\varepsilon|\gamma} = 0 \quad (\text{for } ISTD\Gamma), \quad g_{\beta\varepsilon|\gamma} = 2C_{\beta\varepsilon b}H^{b}_{\gamma} \quad (\text{for } ISTD\Gamma_o), \\
& E^{\varepsilon}_{o\,o} = \overset{o}{E}{}^{\varepsilon}_{o\,o} = 0, \ E^{\varepsilon}_{\beta o} = 0, \ E^{\varepsilon}_{o\gamma} = \overset{o}{E}{}^{\varepsilon}_{\gamma o} = \overset{o}{E}{}^{\varepsilon}_{o\gamma} = C^{\varepsilon}_{\gamma b}H^{b}_{o}, \\
& \tau^{a}_{o\gamma} = -C^{a}_{\gamma b}H^{b}_{o}, \quad \overset{o}{\tau}{}^{a}_{\beta\gamma} = \Gamma^{a}_{\beta\gamma} - \Gamma^{a}_{\gamma\beta} = 0.
\end{aligned} \tag{3.8}
$$

Applying (3.8) to (2.8), we obtain

$$
\widetilde{R}_{oao\delta} = R_{oao\delta} - (C_{a\delta b|\gamma}H^{b}_{o} + C_{a\delta b}H^{b}_{o|\gamma})y^{\gamma} + Z_{a\delta}(\text{or } \overset{o}{Z}_{a\delta}), \tag{3.9}
$$

where

$$
\begin{aligned}
Z_{a\delta} &= C_{a\varepsilon b}H^{b}_{o}T^{\varepsilon}_{\delta} - C_{a\varepsilon b}C^{\varepsilon}_{\delta c}H^{b}_{o}H^{c}_{o} \quad (\text{for } ISTD\Gamma), \\
\overset{o}{Z}_{a\delta} &= 2C_{a\varepsilon b}H^{b}_{o}T^{\varepsilon}_{\delta} + C_{a\varepsilon b}C^{\varepsilon}_{\delta c}H^{b}_{o}H^{c}_{o} \quad (\text{for } ISTD\Gamma_o).
\end{aligned} \tag{3.10}
$$

Since $C_{ijk|\gamma} = C_{ijk|h}B^{h}_{\gamma} + C_{ijk|h}N^{h}_{b}H^{b}_{\gamma}$ and $B^{i}_{\beta|\gamma} = N^{i}_{b}H^{b}_{\beta\gamma}$, because of (3.1) and (3.2) we have

$$
C_{a\delta b|\gamma}y^{\gamma} = P_{a\delta b} + C_{a\delta bc}H^{c}_{o} + C_{abc}H^{c}_{\delta o} + C_{\delta bc}H^{c}_{ao} + C_{a\delta k}N^{k}_{b|\gamma}y^{\gamma}, \tag{3.11}
$$

where $C_{a\delta bc} = C_{ijk|h}B^{ij}_{a\delta}N^{k}_{b}N^{h}_{c}$, $C_{a\delta k} = C_{ijk}B^{ij}_{a\delta}$, $P_{a\delta b} = P_{ijk}B^{ij}_{a\delta}N^{k}_{b}$ and $P^{j}_{i\,k}(=P_{isk}g^{sj})$ is the hv-torsion tensor of Cartan.

It is known [3] that the h-covariant derivative $N^{i}_{b|\gamma}$ is given by

$$
N^{i}_{b|\gamma} = (N^{c}_{j}N^{j}_{b|\gamma})N^{i}_{c} - g^{\beta\varepsilon}(g_{jk|\gamma}B^{j}_{\beta}N^{k}_{b} + \delta_{bc}H^{c}_{\beta\gamma})B^{i}_{\varepsilon}. \tag{3.12}
$$

Since $g_{jk|\gamma} = 0$(or $2C_{jkh}N^h{}_b H^b{}_\gamma$), from (3. 12) we have

$$C_{\alpha\delta k}N^k{}_{b|\gamma}\,y^\gamma = C_{\alpha\delta c}N^c{}_j N^j{}_{b|\gamma}\,y^\gamma - \delta_{bc}C_\alpha{}^\varepsilon{}_\delta H_\varepsilon{}^c{}_o - (2C_\alpha{}^\varepsilon{}_\delta\, C_{\varepsilon bc}H^c{}_o), \tag{3. 13}$$

provided the term in parentheses is valid only for $IST D\Gamma_o$.

Now we shall consider a totally h-auto-parallel subspace M_m of M_n, in which each h-path of M_m with respect to $\widetilde{IM\Gamma}$ is always an h-path with respect to $M\Gamma$. By the use of the methods in [3], from (2. 2) and (2. 3) we can deduce

$$\begin{aligned} Dy^i &= B_\gamma^i Dy^\gamma + N^i{}_b(H^b{}_\gamma du^\gamma), \\ DX^i &= B^i{}_\varepsilon(\widetilde{D}X^\varepsilon - E_\beta{}^\varepsilon{}_\gamma X^\beta du^\gamma) + N^i{}_b(H_\beta{}^b{}_\gamma X^\beta X^\gamma + C_\beta{}^b{}_\gamma X^\beta Dy^\gamma). \end{aligned} \tag{3. 14}$$

In view of (3. 14), we can state.

Proposition 3. 1. *M_m is totally h-auto-parallel with respect to $\widetilde{IM\Gamma}$ if and only if the following equations hold:*

$$E_\beta{}^\varepsilon{}_\gamma + E_\gamma{}^\varepsilon{}_\beta = 0,\quad H^b{}_\gamma = 0,\quad H_\beta{}^b{}_\gamma + H_\gamma{}^b{}_\beta = 0\ (b = m+1, \cdots\cdots, n). \tag{3. 15}$$

Suppose $M\Gamma = STD\Gamma$(or $STD\Gamma_o$). Then from (3. 1) and (3. 2) we have

$$Q_j{}^i{}_k = \Gamma_j{}^i{}_k - \Gamma^i{}_{k\|j} = W_j{}^i{}_k - P_j{}^i{}_k - T_j{}^i{}_k. \tag{3. 16}$$

If $H^b{}_\gamma = 0$, then it follows from (3. 7) and Theorem 7. 2 in [3] that $E_\beta{}^\varepsilon{}_\gamma$(or $\overset{o}{E}_\beta{}^\varepsilon{}_\gamma) = 0$, $H_\beta{}^b{}_\gamma = Q_\beta{}^b{}_\gamma (= Q_j{}^i{}_k N^b{}_i B^{jk}_{\beta\gamma})$ and $T_\beta{}^b{}_\gamma (= T_j{}^i{}_k N^b{}_i B^{jk}_{\beta\gamma}) = 0$. Therefore from (3. 15) and (3. 16) we obtain

$$W_\beta{}^b{}_\gamma (= W_j{}^i{}_k N^b{}_i B^{jk}_{\beta\gamma}) = P_\beta{}^b{}_\gamma\ (= P_j{}^i{}_k N^b{}_i B^{jk}_{\beta\gamma}). \tag{3. 17}$$

Conversely, $H^b{}_\gamma = 0$ and (3. 17) imply (3. 15). Consequently we have

Theorem 3. 2. *M_m is totally h-auto-parallel with respect to $\widetilde{ISTD\Gamma}$ (or $\widetilde{ISTD\Gamma_o}$) if and only if M_m is totally gedesic ($H^h{}_\gamma = 0$) and the equation* (3. 17) *holds.*

Note 3. 1. The above theorem is valid also for $ISTD\Gamma$(or $ISTD\Gamma_o$).

Next we shall consider a totally n-parallel subspace M_m of M_n, in which each normal vector N^i_b is parallel along any curve in M_m with respect to $\widetilde{IM\Gamma}$. The absolute differential $\widetilde{D}N^i_b$ of N^i_b with respect to $\widetilde{IM\Gamma}$ is given by

$$\widetilde{D}N^i_b = N^i{}_{b:\gamma}du^\gamma + N^i{}_{b|\gamma}Dy^\gamma. \tag{3. 18}$$

From (2. 6) we have $N^i{}_{b:\gamma} = N^i{}_{b|\gamma}$. Therefore because of (3. 18), the absolute differential DN^i_b with respect to $IM\Gamma$ is identified with $\widetilde{D}N^i_b$. The theories of such subspaces with respect to $IM\Gamma$ and $\widetilde{I}M\Gamma$ are just the same. The theory with respect to $IM\Gamma$ has been investigated in [4].

§ 4. Semi-induced AMD-connection. An $STD\Gamma$ (or $STD\Gamma_o$) is called an $AMD\Gamma$ (or $AMD\Gamma_o$) if the tensor $T^i{}_k$ in (2. 1) is given by

$$T^i{}_k = f(x, y)\, L(x, y)\, h^i{}_k, \tag{4. 1}$$

where $f(x, y)$ is a $(o)\,p$-homogeneous scalar and $h^i{}_k$ is the angular metric tensor.

For $AMD\Gamma$ (or $AMD\Gamma_o$), from (3. 1), (3. 2) and (4. 1) we first have

$$\Gamma^i{}_k = G^i{}_k + fLh^i{}_k, \quad \Gamma_j{}^i{}_k = \Gamma^*{}_j{}^i{}_k - fLC_j{}^i{}_k. \tag{4. 2}$$

$$Q_j{}^i{}_k = -fLh^i{}_k - P_j{}^i{}_k - T_j{}^i{}_k, \quad D^i{}_k = -fLh^i{}_k, \quad Q_j{}^i{}_o = fLh^i{}_k, \tag{4. 3}$$

where $Q_j{}^i{}_k = \Gamma_j{}^i{}_k - \Gamma^i{}_{k\|j}$ and $T_j{}^i{}_k = (fLh^i{}_k)_{\|j}$.

Further we have

Lemma 4. 1. *The contracted hv-curvature tensors* $P_o{}^i{}_{kh}\,(= y^j P_j{}^i{}_{kh})$ *are given as follows*:

$$P_o{}^i{}_{kh} = P_h{}^i{}_k \ \ (\textit{for } AMD\Gamma), \quad P_o{}^i{}_{kh} = P_h{}^i{}_k + fLC_h{}^i{}_k \ \ (\textit{for } AMD\Gamma_o). \tag{4. 4}$$

Proof. For $AMD\Gamma$, from (4. 2) and (4. 3) we have

$$\begin{aligned} P_o{}^i{}_{kh} &= (\Gamma_j{}^i{}_{k\|h} - C_j{}^i{}_{h|k} - C_j{}^i{}_r Q_h{}^r{}_k)\, y^j \\ &= P_h{}^i{}_k - (fLC_j{}^i{}_k)_{\|h} y^j + C_j{}^i{}_h D^j{}_k = P_h{}^i{}_k. \end{aligned}$$

For $AMD\Gamma_o$, in the similar way we obtain

$$P_o{}^i{}_{kh} = \Gamma_j{}^i{}_{k\|h} y^j = P_h{}^i{}_k - (fLC_j{}^i{}_k)_{\|h} y^j = P_h{}^i{}_k + fLC_h{}^i{}_k. \qquad \text{Q. E. D.}$$

We put

$$T^b{}_\gamma = T^i{}_k N^b{}_i B^k{}_\gamma, \quad D^b{}_\gamma = D^i{}_k N^b{}_i B^k{}_\gamma, \tag{4. 5}$$

$$\overset{b}{H}{}^b{}_\gamma = N^b{}_i (B_o{}^i{}_\gamma + G^i{}_k B^k{}_\gamma), \quad \overset{b}{H}{}_\beta{}^b{}_\gamma = N^b{}_i (B_\beta{}^i{}_\gamma + G_j{}^i{}_k B^{jk}_{\beta\gamma}). \tag{4. 6}$$

For the induced connection $IAMD\Gamma$ (or $IAMD\Gamma_o$), from (1. 5), (1. 6), (1. 11), (4. 1), (4. 3), (4. 5) and (4. 6) we have

$$T^b{}_\gamma = D^b{}_\gamma = Q_\gamma{}^b{}_o = 0, \quad T_\beta{}^b{}_\gamma = 0, \quad H^b{}_\gamma = \overset{b}{H}{}^b{}_\gamma. \tag{4.7}$$

Further from (1.12), (4.1)～(4.3), (4.6) and (4.7) we obtain

$$\begin{aligned} &T^a{}_\gamma = \bar{f}\bar{L}h^a{}_\gamma, \quad Q_\beta{}^b{}_\gamma = -P_\beta{}^b{}_\gamma - \bar{f}\bar{L}C_\beta{}^b{}_\gamma, \quad H_o{}^b{}_\gamma = \overset{b}{H}{}^b{}_\gamma, \\ &H_\beta{}^b{}_o = \overset{b}{H}{}^b{}_\beta + [C_\beta{}^b{}_c\overset{b}{H}{}^c{}_o], \quad H_\beta{}^b{}_\gamma = \overset{b}{H}{}_\beta{}^b{}_\gamma + Q_\beta{}^b{}_\gamma + [C_\beta{}^b{}_c\overset{b}{H}{}^c{}_\gamma], \end{aligned} \tag{4.8}$$

where $\bar{f} = f(x^i(u^\alpha), B^i{}_\alpha y^\alpha)$, $h^a{}_\gamma = h^i{}_k B_i{}^a B^k{}_\gamma$ and the terms in blackets vanish for $IAMD\Gamma_o$.

For the semi-induced connection $\widetilde{I}AMD\Gamma$ (or $\widetilde{I}AMD\Gamma_o$), from (3.5)～(3.7), (4.2) and (4.7) we have

$$\begin{aligned} &\widetilde{\Gamma}_\beta{}^\alpha{}_\gamma = \Gamma_\beta{}^\alpha{}_\gamma + E_\beta{}^\alpha{}_\gamma = \overset{o}{\Gamma}{}_\beta{}^\alpha{}_\gamma + \overset{o}{E}{}_\beta{}^\alpha{}_\gamma, \\ &\Gamma_\beta{}^\alpha{}_\gamma = \overset{o}{\Gamma}{}_\beta{}^\alpha{}_\gamma + C_\beta{}^\alpha{}_b\overset{b}{H}{}^b{}_\gamma, \quad \overset{o}{\Gamma}{}_\beta{}^\alpha{}_\gamma = B_i{}^\alpha(B_\beta{}^i{}_\gamma + \Gamma^*{}_j{}^i{}_k B_\beta{}^{jk}{}_\gamma) - \bar{f}\bar{L}C_\beta{}^\alpha{}_\gamma, \\ &E_\beta{}^\alpha{}_\gamma = C_\gamma{}^\alpha{}_b\overset{b}{H}{}^b{}_\beta - C_{\beta\gamma b}\overset{b}{H}{}^{b\alpha}, \quad \overset{o}{E}{}_\beta{}^\alpha{}_\gamma = E_\beta{}^\alpha{}_\gamma + C_\beta{}^\alpha{}_b\overset{b}{H}{}^b{}_\gamma. \end{aligned} \tag{4.9}$$

By virtue of Theotem 3.2 and (4.8) we can state

Proposition 4.1. *Let $J\Gamma$ be any one of the four connections $IAMD\Gamma$, $IAMD\Gamma_o$, $\widetilde{I}AMD\Gamma$ and $\widetilde{I}AMD\Gamma_o$. Then M_m is totally h-auto-parallel with respect to $J\Gamma$ if and only if M_m is totally geodesic and the following equation holds* :

$$P_\beta{}^b{}_\gamma + \bar{f}\bar{L}C_\beta{}^b{}_\gamma = 0. \tag{4.10}$$

Let M_n be endowed with an $AMD\Gamma$ (or $AMD\Gamma_o$) and a subspace M_m of M_n be totally *ncd*-free(or *nc*-constant). Then there exist n-m direct-free scalars $f^b(u^\beta, y^\beta)$ ($f^b{}_{|\gamma} = 0$) (or constants f^b) such that

$$\overset{b}{H}{}^b{}_o = \bar{L}^2 f^b \ (b = m+1, \cdots\cdots, n). \tag{4.11}$$

In this case, the square of the normal curvature $N(u^\beta, y^\beta)$ in y^β-direction at each point (u^β) is given by $N^2 = \delta_{bc} f^b f^c$.

The condition (4.11) implies the following two equivalent ones:

$$\overset{b}{H}{}^b{}_\gamma = f^b y_\gamma - \frac{1}{2}\bar{L}^2\lambda_c{}^b{}_\gamma f^c, \quad y_\gamma := g_{\beta\gamma}y^\beta, \tag{4.12}$$

$$\overset{b}{H}{}_\beta{}^a{}_\gamma = f^a g_{\beta\gamma} - f^b(\lambda_b{}^a{}_\beta y_\gamma + \lambda_b{}^a{}_\gamma y_\beta) - \frac{1}{2}\bar{L}^2 f^b(\lambda_b{}^a{}_{\gamma\|\beta} - \lambda_c{}^a{}_\beta\lambda_b{}^c{}_\gamma). \tag{4.13}$$

Applying (4.4), (4.7), (4.8) and (4.11)～(4.13) to (1.13), we have

(4. 14) $$\frac{1}{2}(R_{o\gamma o\delta}+R_{o\delta o\gamma})=\frac{1}{2}(R_{oioh}+R_{ohoi})B^{ih}_{\gamma\delta}+\bar{L}^2N^2h_{\gamma\delta}+\frac{1}{2}(U_{\gamma\delta}+V_{\gamma\delta})$$

where

(4. 15) $$U_{\gamma\delta}=\bar{L}^2\Big\{\bar{L}^2\big(\sum_a f^a\lambda^a_{b\gamma}\lambda^b_{c\delta}f^c-C_{ab\delta\|\gamma}f^af^b-\frac{1}{2}\sum_a\lambda^a_{b\gamma}\lambda^a_{c\delta}f^bf^c\big)-(C_{ab\gamma}y_\delta+C_{ab\delta}y_\gamma)f^af^b\Big\},$$

(4. 16) $$V_{\gamma\delta}=-4\bar{L}^2(\bar{f}\bar{L}C_{\gamma\delta a}+P_{\gamma\delta a})f^a+[2\bar{L}^3\bar{f}C_{\gamma\delta a}f^a],$$

where the term in blackets vanishes for $IAMD\Gamma_o$.

We impose the following assumption:

(4. 17) $$\sum_b\lambda^b_{c\gamma}f^b=0,\quad \lambda^b_{c\gamma}f^c=0.$$

If we put $\bar{N}^i=N^i_bf^b$, then the above assumption means

(4. 18) $$g_{ij}N^i_{b\|\gamma}\bar{N}^j=0,\quad g_{ij}N^i_b\bar{N}^j{}_{\|\gamma}=0.$$

From (1. 6) and (4. 17) we have

(4. 19) $$C_{ab\gamma}f^a=0.$$

We shall call an assunption (4. 17) the *mixed normality condition* (or simply *mn-condition*) *with scalars* f^b.

In this case, it is easily seen that the tensor $U_{\gamma\delta}$ in (4. 15) vanishes. Consequently we can state.

Proposition 4. 2. *Suppose that* M_n *is endowed with an* $AMD\Gamma$ (or $AMD\Gamma_o$) *and a subspace* M_m *of* M_n *is totally ncd-free* (*or nc-constant*) *with* $N^2=\delta_{bc}f^bf^c$, *and that the normals satisfy the mn-codition* (4. 17) *with* f^b *and the tensor* $V_{\gamma\delta}$ *in* (4. 16) *vanishes. Then if* M_n *is of scalar curvature* R (*resp. constant curvature* R) *with respect to* $AMD\Gamma$ (*or* $AMD\Gamma_o$), *then* M_m *is also of scalar curvature* $(R+N^2)$ (*resp. constant curvature* $(R+N^2)$ *with respect to* $IAMD\Gamma$ (*or* $IAMD\Gamma_o$).

Next we can state

Lemma 4. 2. *If the normals satisfy the mn-condition* (4. 17) *with direc-free scalars* f^b, *then the following relation holds*:

(4. 20) $$C_{a\delta bc}f^c=2(C_{a\varepsilon c}C^\varepsilon_{\delta b}+C_{a\varepsilon b}C^\varepsilon_{\delta c})f^b-[(C_{a\varepsilon c}C^\varepsilon_{\delta b}+C_{a\varepsilon b}C^\varepsilon_{\delta c})f^b],$$

where the terms in blackets vanish for $IAMD\Gamma_o$.

Proof. Differentiating (4. 19) partially by y^δ, we have

$$(C_{bca}f^b)_{\|\delta} = (C_{ijk\|h}B^h{}_\delta N^i{}_b N^j{}_c + C_{ijk}N^i{}_{b\|\delta}N^j{}_c + C_{ijk}N^i{}_b N^j{}_{c\|\delta})B^k{}_a f^b = 0. \tag{4.21}$$

Applying (1. 6) to (4. 21) and using (4. 17), (4. 19) and the symmetric relation $C_{ijk\|h} = C_{khi\|j}$, for $IAMD\Gamma_o$ we first obtain

$$C_{khi\|j}B^{kh}_{a\delta}\ N^i{}_b N^j{}_c f^b = 2(C_{a\varepsilon c}C^\varepsilon_{\delta}{}_b + C_{a\varepsilon b}C^\varepsilon_{\delta}{}_c)f^b. \tag{4.22}$$

Since $B^i_\varepsilon B^\varepsilon{}_r = \delta^i{}_r - N^i{}_d N^d{}_r$, by the use of (4. 19) we have

$$\begin{aligned} C_{\varepsilon\delta d}C^\varepsilon_a{}_c f^b &= C_{i\delta b}C^r_a{}_c(\delta^i{}_r - N^i{}_d N^d{}_r)f^b \\ &= C_{r\delta b}C^r_a{}_c f^b - C_{d\delta b}C^d_a{}_c f^b = C_{r\delta b}C^r_a{}_c f^b. \end{aligned} \tag{4.23}$$

For $IAMD\Gamma$, from (4. 19), (4. 22) and (4. 23) we obtain

$$C_{khi}|_j B^{kh}_{a\delta}\ N^i{}_b N^j{}_c f^b = (C_{a\varepsilon c}C^\varepsilon_\delta{}_b + C_{a\varepsilon b}C^\varepsilon_\delta{}_c)f^b. \tag{4.24}$$

Thus from (4. 22) and (4. 24) we have (4. 20). Q. E. D.

From (3. 9)～(3. 11), (3. 13), (4. 8), (4. 11), (4. 12), (4. 17), (4. 19) and (4. 20) we can deduce

$$\begin{aligned} \frac{1}{2}(\widetilde{R}_{\alpha\gamma\alpha\delta} + \widetilde{R}_{\alpha\delta\alpha\gamma}) &= \frac{1}{2}(R_{\alpha\gamma\alpha\delta} + R_{\alpha\delta\alpha\gamma}) - \bar{L}^2\{(P_{\gamma\delta b} - 2\bar{f}\bar{L}C_{\gamma\delta b})f^b \\ &+ 3\bar{L}^2 C_{\gamma\delta b}C^\varepsilon_\delta{}_c f^b f^c + C_{\gamma\delta c}N^c{}_j \bar{N}^j{}_{|\gamma}y^\gamma\} - [\bar{L}^3\bar{f}C_{\gamma\delta b}f^b], \end{aligned} \tag{4.25}$$

where the term in blackets vanishes for $IAMD\Gamma_o$.

Applying (4. 14)～(4. 16) to (4. 25), we obtain

$$\begin{aligned} \frac{1}{2}(\widetilde{R}_{\alpha\gamma\alpha\delta} + \widetilde{R}_{\alpha\delta\alpha\gamma}) &= \frac{1}{2}(R_{oioh} + R_{ohoi})B^{ih}_{\gamma\delta} + \bar{L}^2 N^2 h_{\gamma\delta} \\ &- \bar{L}^2\{3(P_{\gamma\delta b}f^b + \bar{L}^2 C_{\gamma\delta b}C^\varepsilon_\delta{}_c f^b f^c) + C_{\gamma\delta c}N^c{}_j\bar{N}^j{}_{|\gamma}y^\gamma\}. \end{aligned} \tag{4.26}$$

Now we consider the following differential equation:

$$3(P_{\gamma\delta b}f^b + \bar{L}^2 C_{\gamma\varepsilon b}C^\varepsilon_\delta{}_c f^b f^c) + C_{\gamma\delta c}N^c{}_j\bar{N}^j{}_{|\gamma}y^\gamma = 0. \tag{4.27}$$

Then in view of (4. 26) we can state

Theorem 4. 3. *Suppose that M_n is endowed with an $AMD\Gamma$ (or $AMD\Gamma_o$) and a subspace M_m of M_n is totally ncd-free (resp. nc-constant) with $N^2 = \delta_{bc}f^b f^c$, and that the normals are chosen in such ways that they satisfy the equation* (4. 27) *and the mn-condition* (4. 17) *with* f^b. *Then if* M_n *is of scalar curvature* R (*resp. constant curvature* R) *with respect to* $AMD\Gamma$ (*or* $AMD\Gamma_o$), *then* M_m *is also of scalar curvature* $(R+N^2)$ (*resp. constant curvature* $(R+N^2)$ *with respect to* $\widetilde{T}AMD\Gamma$ (*or* $\widetilde{T}AMD\Gamma_o$).

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