Ann. Rep. Asahikawa Med. Coll.
1986. Vol.7, 1～17

A STUDY OF CONNECTIONS IN A FINSLER SPACE

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Introduction. The theories of connections on Finsler spaces have been studied by many authors from their own standpoints. As well-known connections, there are, above all, the Berwald connection, the Cartan and the Rund etc.. Geometrical properties with respect to these connections are usually discussed. These connections commonly reduce to the Riemannian ones when Finsler spaces become the Riemannian spaces. And with respect to these connections, the deflexion tensors commonly vanish. Here we have a question if it is necessary to consider connections with deflexion tensors in Finsler geometry.

The princrpal purpose of the present paper is to find such connections that the following conditions are satisfied.

(1) They reduce to the Riemannian connections when Finsler spaces become the Riemannian spaces.

(2) Their deflexion tensors do not vanish.

(3) They are closely similar to the above connections, that is, they satisfy as many axioms as the well-known connections, except (2), satisfy.

(4) They are expressible in as simple forms as possible.

For this, we shall firstly consider connections on Finsler spaces from the most general standpoint. This is done in § 1. Secondly we shall consider them from a view-point of axiomatic theory. This is done in § 2. In § 3, we shall lastly consider our problem.

The terminologies and notations refer to the papers [8] – [10] unless otherwise stated.

§ 1. Matsumoto connections and Kawaguchi connections. Let M be an n-dimentional Finsler space with a fundamental function $L(x, y)$. Then we can define various connections on M. Now we shall consider one of the most general connections on M. The connection may be represented by

$$(1.1) \qquad \Gamma=(\Gamma_j{}^i{}_k,\ \Gamma^i{}_k,\ \widetilde{C}_j{}^i{}_k)\quad [5]^{1)},$$

where $\Gamma_j{}^i{}_k$, $\Gamma^i{}_k$ and $\widetilde{C}_j{}^i{}_k$ are positively homogeneous of degree 0, 1 and −1 in y^i respectively, and are called the h-connection, non-linear connection and v-connection of Γ respectively.

In the following, we shall simply say that a quantity $f(x, y)$ on M is $(r)p$-homogeneous if it is positively homogeneous of degree r in y^i.

The hv-torsion tensor $\widetilde{P}^i{}_{kj}$ with respect to Γ is defined by

$$(1.2) \qquad \widetilde{P}^i{}_{kj}=\Gamma^i{}_{k\|j}-\Gamma_j{}^i{}_k,$$

where the symbol $\|$ denotes the partial differentiation by y^j. Therefore if we put $Q_j{}^i{}_k=-\widetilde{P}^i{}_{kj}$, then the h-connection is expressible in

$$(1.3) \qquad \Gamma_j{}^i{}_k=\Gamma^i{}_{k\|j}+Q_j{}^i{}_k,$$

where $Q_j{}^i{}_k$ is a $(o)p$-homogeneous tensor.

If we denote the non-linear connection of Cartan (or Berwald) by $G^i{}_k$, then the non-linear connection $\Gamma^i{}_k$ is expressible in

$$(1.4) \qquad \Gamma^i{}_k=G^i{}_k+T^i{}_k$$

for a $(1)p$-homogenaous tensor $T^i{}_k$. Applying (1.4) to (1.3), we have

$$(1.5) \qquad \Gamma_j{}^i{}_k=G_j{}^i{}_k+T_j{}^i{}_k+Q_j{}^i{}_k,$$

where $G_j{}^i{}_k(=G^i{}_{k\|j})$ is the h-connection of Berwald and $T_j{}^i{}_k=T^i{}_{k\|j}$.

Let three tensors $T^i{}_k$, $Q_j{}^i{}_k$ and $\widetilde{C}_j{}^i{}_k$ be given as follows:

(1.6)
1) $T^i{}_k$ is a $(1)p$-homogeneous (1,1)-tensor.
2) $Q_j{}^i{}_k$ is a $(0)p$-homogeneous (1,2)-tensor.

1) Numbers in brackets refer to the references at the end of the paper.

3) $\widetilde{C}_j{}^i{}_k$ is a $(-1)p$-homogeneous (1,2)-tensor.

Then a connection Γ on M is uniquely determined by (1.1), (1.4), (1.5) and (1.6). We shall call the above connection Γ a *Matsumoto connection*[2)] on a Finsler space M and denote it by $M\Gamma$. M. Matsumoto investigated various connections on M, introduced the most general ones and established a theory of connections on M[5].

Contracting (1.3) by y^j, we have

$$(1.7) \qquad D^i{}_k := y^j \Gamma_j{}^i{}_k - \Gamma^i{}_k = Q_o{}^i{}_k,$$

where $D^i{}_k$ is called the deflexion tensor and $Q_o{}^i{}_k = y^j Q_j{}^i{}_k$.

From (1.7) we can state

Lemma 1.1. *For an $M\Gamma$, the deflexion tensor vanishes if and only if the tensor $Q_j{}^i{}_k$ is indicatric in the index j, namely $Q_o{}^i{}_k = 0$.*

Note 1.1. Well-known connections, for example the Berwald connection $B\Gamma$ and the Cartan connection $C\Gamma$, have no deflexion tensor, namely always $Q_o{}^i{}_k = 0$.

From (1.4) it follows that the h-covariant derivative $L_{|k}$ is given by

$$(1.8) \qquad L_{|k} = \delta_k L = \partial L / \partial x^k - \Gamma^j{}_k L_{\|j} = -T^o{}_k / L.$$

We shall say that an $M\Gamma$ is metrical if $L_{|k} = 0$. In this case, because of (1.8) we can state

Lemma 1.2. *An $M\Gamma$ is metrical if and only if the tensor $T^i{}_k$ indicatric in the upper index i, namely $T^o{}_k = 0$.*

Note 1.2. If we consider tangent spaces at every points of M as Minkowski spaces, then indicatrics at every points correspond mutually to themselves under the metrical $M\Gamma$, that is, the lengths of tangent vectors are invariant under any parallel displacement with respect to this connection [8].

Contracting (1.5) by $y^j y^k$, we have

$$(1.9) \qquad \Gamma_j{}^i{}_k y^j y^k = 2G^i + T^i{}_o + Q_o{}^i{}_o.$$

Because of (1.9) we can state

2) This connection is called a Finsler one in [5], but it seems to us that our nomenclature is preferable.

Lemma 1 . 3 . *Paths with respect to an $M\Gamma$ are always geodesics of M if and only if the following equation holds:*

$$(1.10) \qquad T^i{}_o + Q_o{}^i{}_o \; (= D^i{}_o) = 0.$$

The fundamental tensor g_{ij} is given by $g_{ij} = \frac{1}{2} L^2{}_{\cdot i \cdot j}$ and the v-torsion tensor and hv-torsion tensor with respect to $C\Gamma$ are as follows:

$$C_{ijk} := \tfrac{1}{2} g_{ij \cdot k}, \quad P_{ijk} := C_{ijk;h} y^h,$$

where the semi-colon ; indicates the h-convariant differentiation of Cartan.

The h-convariant and v-convariant derivatives of g_{ij} are respectively given by

$$(1.11) \qquad g_{ij|k} = -\{ T_{ijk} + T_{jik} + Q_{ijk} + Q_{jik} + 2(C_{ijr} T^r{}_k + P_{ijk}) \},$$

$$(1.12) \qquad g_{ij}|_k = 2C_{ijk} - \widetilde{C}_{ijk} - \widetilde{C}_{jik},$$

where $T_{ijk} = g_{jr} T_i{}^r{}_k$, $Q_{ijk} = g_{jr} Q_i{}^r{}_k$ and $\widetilde{C}_{ijk} = g_{jr} \widetilde{C}_i{}^r{}_k$.

We shall say that an $M\Gamma$ is h-metrical or v-metrical according to whether $g_{ij|k} = 0$ or $g_{ij}|_k = 0$. Then from (1.11) and (1.12) we can state

Lemma 1 . 4 . *An $M\Gamma$ is h-metrical or v-metrical if and only if the following equation* (1.13) *or* (1.14) *holds respectively:*

$$(1.13) \qquad T_{ijk} + T_{jik} + Q_{ijk} + Q_{jik} + 2(C_{ijr} T^r{}_k + P_{ijk}) = 0,$$

$$(1.14) \qquad 2C_{ijk} = \widetilde{C}_{ijk} + \widetilde{C}_{jik}.$$

If we put $T_{jk} = g_{jr} T^r{}_k$, then from (1.5) we have

$$(1.15) \qquad T_j{}^i{}_k y_i = T_{jok} = T^i{}_{k \cdot j} y_i = T^o{}_{k \cdot j} - T_{jk}.$$

If an $M\Gamma$ is metrical, then from (1.5) and Lemma 1.2 we obtain

$$(1.16) \qquad T_j{}^i{}_k y_i \; (= T_{jok}) = - T_{jk}.$$

Conversely if the equation (1.16) holds, then from (1.5) we have $T^o{}_{k \cdot j} = 0$. Contracting this result by y^j, we have $2\, T^o{}_k = 0$. Consequently we can state

Proposition 1 . 1 . *The following three facts are mutually equivalent:*

(1) *An $M\Gamma$ is metrical.* (2) $T^o{}_k = 0$.

(3) *An equation* (1.16) *holds.*

Contracting (1.13) by y^i and using (1.15) we have

$$T^o{}_{k\|j} + Q_{ojk} + Q_{jok} = 0. \tag{1.17}$$

In this case, we obtain $T^o{}_k = 0$ if the following equation holds:

$$Q_{ojk} + Q_{jok} = 0. \tag{1.18}$$

Note 1.3. An $M\Gamma$ is not, in general, metrical even if it is h-metrical. If the $M\Gamma$ is h-metrical and satisfies (1.18), then it is metrical.

The absolute differential Dg_{ij} *of* g_{ij} is given by

$$Dg_{ij} = g_{ij|k}\, dx^k + g_{ij}|_k\, Dy^k, \quad Dy^k = dy^k + \Gamma^k{}_h\, dx^h. \tag{1.19}$$

Contracting (1.14) by y^i, we have

$$\widetilde{C}_{ojk} + \widetilde{C}_{jok} = 0. \tag{1.20}$$

It follows from (1.13), (1.14), (1.16) and (1.19) that we have always $y^i Dg_{ij} = 0$ if and only if equations (1.17) and (1.20) hold.

Since $y_j = g_{ij}\, y^i$, a relation $y^i Dg_{ij} = 0$ is equivalent to $Dy_j = g_{ij} Dy^i$. Therefore we can state

Lemma 1.5. *With respect to an* $M\Gamma$, *the absolute differential* Dy_j *of* y_j *is given by*

$$Dy_j = g_{ij}\, Dy^i \quad (\text{or equivalently } y^i Dg_{ij} = 0) \tag{1.21}$$

if and only if equations (1.17) and (1.20) hold.

Note 1.4. A. Kawaguchi imposed an assumption (1.21) on the connection in consideration in his theory of non-linear connections [4]. This assumption also enables us to define the angle between vectors y^i and X^i.

We shall say that an $M\Gamma$ is v-symmetric if the v-connection is symmetric, namely $\widetilde{C}_j{}^i{}_k = \widetilde{C}_k{}^i{}_j$. Then from this defintion and (1.14) we can state

Lemma 1.6. *An* $M\Gamma$ *is both* v*-metrical and* v*-symmetric if and only if* $\widetilde{C}_j{}^i{}_k = C_j{}^i{}_k \,(= g^{ir} C_{jrk})$.

An $M\Gamma = (\Gamma_j{}^i{}_k,\ \Gamma^i{}_k,\ \widetilde{C}_j{}^i{}_k)$ has no metrical property. However we can easily construct a connection $\Gamma=(\overline{\Gamma}_j{}^i{}_k,\ \overline{\Gamma}^i{}_k,\ \overline{C}_j{}^i{}_k)$ which has all metrical properties. Firstly we take

$$(1.22)\qquad \overline{\Gamma}^i{}_k=G^i{}_k+\overline{T}^i{}_k,\ \ \overline{T}^i{}_k=f(x,y)h^i{}_r\,T^r{}_k,$$

where $h^i{}_r\ (=\delta^i{}_r-l^i\,l_r)$ is the angular metric tensor and $f(x, y)$ is a $(0)p$-homogeneous scalar. Secondly if we take

$$(1.23)\qquad \overline{C}_j{}^i{}_k=\widetilde{C}_j{}^i{}_k+\tfrac{1}{2}g^{ir}g_{rj}|_k,$$

then (1. 23) is, because of (1.12), expressible in

$$(1.24)\qquad \overline{C}_j{}^i{}_k=C_j{}^i{}_k+\tfrac{1}{2}(\widetilde{C}_j{}^i{}_k-\widetilde{C}^i{}_{jk}),$$

where $\widetilde{C}^i{}_{jk}=g^{ir}\,\widetilde{C}_{rjk}$. Lastly we put

$$(1.25)\qquad \widetilde{\Gamma}_j{}^i{}_k=G_j{}^i{}_k+\overline{T}_j{}^i{}_k+Q_j{}^i{}_k,\ \ \overline{T}_j{}^i{}_k=\overline{T}^i{}_{k\|j}$$

and take

$$(1.26)\qquad \overline{\Gamma}_j{}^i{}_k=\widetilde{\Gamma}_j{}^i{}_k+\tfrac{1}{2}g^{ir}g_{rj|k},$$

where the symbol | means the h-covariant differentiation with respect to $\overline{\Gamma}^i{}_k$ and $\widetilde{\Gamma}_j{}^i{}_k$. From (1.11), (1.22), (1.25) and (1.26) we have

$$(1.27)\qquad \overline{\Gamma}_j{}^i{}_k=\Gamma^*{}_j{}^i{}_k+\tfrac{1}{2}(\overline{T}_j{}^i{}_k-\overline{T}^i{}_{jk})+\tfrac{1}{2}(Q_j{}^i{}_k-Q^i{}_{jk})-C_j{}^i{}_k\,\overline{T}^r{}_k,$$

where $\overline{T}^i{}_{jk}=g^{ir}\,\overline{T}_{rjk}$, $Q^i{}_{jk}=g^{ir}\,Q_{rjk}$ and $\Gamma^*{}_j{}^i{}_k$ is the h-connection of Cartan.

In this case, we can easily prove that the connection $\overline{\Gamma}=(\overline{\Gamma}_j{}^i{}_k,\ \overline{\Gamma}^i{}_k,\ \overline{C}_j{}^i{}_k)$ obtained thus is metrical, h-metrical and v-metrical.

We shall call the above connection $\overline{\Gamma}$ a *Kawaguchi connection* of an $M\Gamma$ and denote it by $K(M\Gamma)$.

Note 1 . 5 . Metrizations (1.23) and (1.26) are due to A. Kawaguchi [3]. $B\Gamma$ is metrical but neither h-metrical nor v-metrical. The Rund connection $R\Gamma$ is both metrical and h-metrical but not v-metrical. The Hashiguchi connection $H\Gamma$ is both metrical and v-metrical but not h-metrical. In this case, any of $K(B\Gamma)$, $K(R\Gamma)$ and $K(H\Gamma)$ becomes $C\Gamma$.

§ 2 . ***TM*-connections.** First we shall give important axioms concerning connections in Finsler geometry. Let $\Gamma=(\Gamma_j{}^i{}_k,\ \Gamma^i{}_k,\ \widetilde{C}_j{}^i{}_k)$ be the connection in consideration. Then axioms are as follows:

(F1) Γ is metrical, i.e. $L_{i|k}=0$.

(F2) The deflexion tensor $D^i{}_k$ of Γ vanishes.

(F3) Γ is v-metrical and v-symmetric.

(F3)$_1$ The v-connection of Γ vanishes, i.e. $\tilde{C}_j{}^i{}_k=0$.

(F3)$_2$ The v-connection of Γ is semi-symmetric, that is

$$\tilde{C}_j{}^i{}_k-\tilde{C}_k{}^i{}_j=\delta^i{}_j\, t_k-\delta^i{}_k\, t_j\ , \tag{2.1}$$

where t_k is a $(-1)p$-homogeneous vector.

(F4) With respect to Γ, the absolute differential Dy_i of $y_i\,(=g_{ij}\,y^j)$ is given by $Dy_i=g_{ij}\,Dy^j$ (or equivalently $y^i Dg_{ij}=0$).

(F5) Paths with respect to Γ are always geodesics of M.

(F6) Γ is h-metrical.

(F7) Γ is h-symmetric, that is, the h-torsion tensor $\tau_j{}^i{}_k\,(=\Gamma_j{}^i{}_k-\Gamma_k{}^i{}_j)$ vanishes.

(F8) The hv-torsion tensor $\tilde{P}^i{}_{kj}\ (=-Q_j{}^i{}_k)$ of Γ vanishes.

(F9) The h-connection of Γ is semi-symmetric, that is,

$$\tau_j{}^i{}_k=\delta^i{}_j\, s_k-\delta^i{}_k\, s_j\ , \tag{2.2}$$

where s_k is a $(0)p$-homogeneous vector.

A Matsumoto connection $M\Gamma$ is called a TM(resp. $TM(0)$)-connection if it is characterized by five axioms (F1), (F2), (F3) (resp. (F3)$_1$), (F4) and (F5). From Lemmas 1.1～1.3, Lemmas 1.5 and 1.6 we can state

Theorem 2.1. *A TM (resp. $TM(0)$)-connection is uniquely determined if three tensors $T^i{}_k$, $Q_j{}^i{}_k$ and $\tilde{C}_j{}^i{}_k$ are given as follows:*

1) *$T^i{}_k$ is a $(1)p$-homogeneous (1,1)-tensor such that $T^o{}_k=T^i{}_o=0$.*

2) *$Q_j{}^i{}_k$ is a $(0)p$-homogeneous (1,2)-tensor such that $Q_j{}^o{}_k=Q_o{}^i{}_k=0$.*

3) *$\tilde{C}_j{}^i{}_k=C_j{}^i{}_k$ (resp. 0)*

Immediately we have

Corollary 2.1.1. *Typical four connections $C\Gamma$, $H\Gamma$, $R\Gamma$ and $B\Gamma$ are special TM (or $TM(0)$)-connections, and their characterizing axioms and three tensors are as follows :*

Connections	Axioms	Tensors $(T, Q, \widetilde{C})$
$C\Gamma$	(F2), (F3), (F6), (F7)	$T^i{}_k=0$, $Q_j{}^i{}_k=-P_j{}^i{}_k$, $\widetilde{C}_j{}^i{}_k=C_j{}^i{}_k$
$R\Gamma$	(F2), $(F3)_1$, (F6), (F7)	$T^i{}_k=0$, $Q_j{}^i{}_k=-P_j{}^i{}_k$, $\widetilde{C}_j{}^i{}_k=0$
$H\Gamma$	(F1), (F2), (F3), (F7), (F8)	$T^i{}_k=0$, $Q_j{}^i{}_k=0$, $\widetilde{C}_j{}^i{}_k=C_j{}^i{}_k$
$B\Gamma$	(F1), (F2), $(F3)_1$, (F7), (F8)	$T^i{}_k=0$, $Q_j{}^i{}_k=0$, $\widetilde{C}_j{}^i{}_k=0$

It is known that M is a Riemannian space $\overset{r}{M}$ if and only if the torsion tensor C_{ijk} vanishes. In this case, the foundamental function $L(x, y)$ is given by $L(x, y)=(g_{ij}(x)y^i y^j)^{\frac{1}{2}}$ and the Riemannian connection is, from the standpoint of Finsler geometry, represented by

$$(2.3)\qquad RN\Gamma=(\{_j{}^i{}_k\},\ y^j\{_j{}^i{}_k\},\ 0),$$

where $\{_j{}^i{}_k\}$ are the Christoffel symbols formed with $g_{ij}(x)$.

Note 2.1. A Riemannian space $\overset{r}{M}$ may be considered as a special Finsler space. There however exists some Finsler space such that it never reduces to an $\overset{r}{M}$ [5].

Now we consider the following axiom :

(F0) The fundamental tensor g_{ij} is independent of y^i, i.e. $C_{ijk}=0$.

Then we can state

Theorem 2.2. *An $M\Gamma$ is the $RN\Gamma$ if and only if five axioms* (F0), (F2), $(F3)_1$, (F6) *and* (F7) *are satisfied.*

It is easily seen that if M becomes an $\overset{r}{M}$, then any of typical connections in Corollary 2.1.1 reduces to $RN\Gamma$.

Note 2.2. With respect to $RN\Gamma$, axioms (F1) and (F6) are mutually equivalent.

A TM (or $TM(0)$)-connection is called a TMA (or $TMA(0)$)-connection if the $T^i{}_k$ is given by

$$(2.4)\qquad T^i{}_k=f(x, y)\,L(x, y)\,h^i{}_k,$$

where $f(x, y)$ is a $(0)p$-homogeneous scalar. Differentiating (2.4) by y^j, we have

$$(2.5)\qquad T_j{}^i{}_k=Lf_{|j}\,h^i{}_k+f(l_j h^i{}_k-l_k h^i{}_j-l^i h_{jk}).$$

We know already four typical TMA (or $TMA(0)$)-connections, that is, AMN

Γ, $AMB\Gamma$, $AMC\Gamma$ and $AMR\Gamma$ (or respective (0)-connections) [9]. For these connections, the tensors $Q_j{}^i{}_k$ are respectively given as follows :

$$\text{(2.6)}\quad \begin{array}{ll} 1)\ -Lf_{\|j}\ h^i{}_k, & 2)\ 2fl_k h^i{}_j - Lf_{\|j}\ h^i{}_k, \\ 3)\ 2fl_k h^i{}_j - Lf_{\|j} h^i{}_k - P_j{}^i{}_k, & 4)\ -Lf_{\|j} h^i{}_k + fl_k h^i{}_j - fLC_j{}^i{}_k - P_j{}^i{}_k. \end{array}$$

Therefore the respective h-connections are expressed as follows :

$$\text{(2.7)}\quad \begin{array}{l} 1)\ \Gamma_j{}^i{}_k = G_j{}^i{}_k + f(l_j\ h^i{}_k - l_k h^i{}_j - l^i h_{jk}), \\ 2)\ \Gamma_j{}^i{}_k = G_j{}^i{}_k + f(l_j\ h^i{}_k + l_k h^i{}_j - l^i h_{jk}), \\ 3)\ \Gamma_j{}^i{}_k = \Gamma^*{}_j{}^i{}_k + f(l_j\ h^i{}_k + l_k h^i{}_j - l^i h_{jk}), \\ 4)\ \Gamma_j{}^i{}_k = \Gamma^*{}_j{}^i{}_k + f(l_j\ h^i{}_k - l^i h_{jk} - LC_j{}^i{}_k). \end{array}$$

It is easily seen that an $AMR\Gamma$ (the forth connection) is metrical, h-metrical and v-metrical. Further we can state

Thorem 2 . 3 . *Let Γ be any connection of seven connections ($AMN\Gamma$, $AMN\Gamma(0)$, $AMB\Gamma$, $AMB\Gamma(0)$, $AMC\Gamma$, $AMC\Gamma(0)$, $AMR\Gamma(0)$). Then a Kawaguchi connection $K(\Gamma)$ becomes an $AMR\Gamma$.*

Proof. Since any TMA-connection is metrical, it is enough to consider the same non-linear connection. From (2.5) we have

$$\text{(2.8)}\qquad T_j{}^i{}_k - T^i{}_{jk} = -L(f_{\|j}\ h^i{}_k - g^{ir} f_{\|r} h_{jk}) + 2f(l_j\ h^i{}_k - l^i h_{jk}).$$

In every case of (2.6), we have

$$\text{(2.9)}\qquad Q_j{}^i{}_k - Q^i{}_{jk} = -L(f_{\|j}\ h^i{}_k - g^{ir} f_{\|r} h_{jk}).$$

Therefore it follows from (1.27), (2.8) and (2.9) that the h-connection of $K(\Gamma)$ coincides with 4) in (2.7). For the v-connection, from (1.24) we obtain $\overline{C}_j{}^i{}_k = C_j{}^i{}_k$ in every case. Q. E. D.

For an $AMR\Gamma$, we can state

Theorem 2 . 4 . *An $M\Gamma$ is an $AMR\Gamma$ (resp. $AMR\Gamma(0)$) if and only if the five axioms* (F2), (F3) (resp. $(\mathrm{F3})_1$), (F5), (F6) *and* (F9) *are satisfied.*

Proof. If an $M\Gamma$ is an $AMR\Gamma$ (resp. $AMR\Gamma(0)$), then we can first verify that the four axioms (F2), (F3) (resp. $(\mathrm{F3})_1$), (F5) and (F6) are satisfied. Next if we put $s_k = -f l_k$, then from 4) in (2.7) we obtain

$$\tau_j{}^i{}_k = f(l_j h^i{}_k - l_k h^i{}_j) = \delta^i{}_j s_k - \delta^i{}_k s_j ,$$

which shows that the last axiom (F9) is satisfied.

Conversely suppose that four axioms (F2), (F3) (resp. $(F3)_1$), (F6) and (F9) are first satisfied. Then from Theorem 2.2 in [10] and Lemma 1.6 we have

(2. 10) $\qquad \widetilde{C}_j{}^i{}_k = C_j{}^i{}_k \ (resp.\ 0)$,

(2. 11) $\qquad T^i{}_k = s^i y_k - \delta^i{}_k s_o - L^2 C^i_{k\,r} s^r$,

(2. 12)
$$\Gamma_j{}^i{}_k = \Gamma *_j{}^i{}_k + g_{j\,k} s^i - s_j \delta^i{}_k + C_j{}^i{}_k s_o + (y^i C_{j\,kr} - C^i_{k\,r} y_j) s^r$$
$$+ L^2 (C_j{}^i{}_r C^r_{k\,t} + C^i_{k\,r} C_j{}^r{}_t - C^i_{r\,t} C_j{}^r{}_k) s^t .$$

Next let the remaining axioms (F5) be satisfied. Contracting (2.12) by $y^j y^k$, we have $\Gamma_j{}^i{}_k y^j y^k = 2G^i + L^2 s^i - s_o y^i$. Therefore from (1.9) and Lemma 1. 3 we obtain $L^2 s^i - s_o y^i = 0$, from which it follows that

(2. 13) $\qquad s^i = -f l^i , \ s_i = -f l_i$,

where $f = -s_o / L$. Applying (2.13) to (2.11) and (2.12), we obtain

(2. 14) $\qquad T^i{}_k = fLh^i{}_k , \ \Gamma^i{}_k = -G^i{}_k + fLh^i{}_k$,

(2. 15)
$$\Gamma_j{}^i{}_k = \Gamma *_j{}^i{}_k + f(l_j \delta^i{}_k - l^i g_{j\,k} - LC_j{}^i{}_k)$$
$$= \Gamma *_j{}^i{}_k + f(l_j h^i{}_k - l^i h_{j\,k} - LC_j{}^i{}_k) .$$

Thus it follows from (2.10), (2.14) and (2.15) that this connection is an *AMR* Γ (resp. *AMR*Γ(0)). Q. E. D.

Note 2 . 3 . An *M*Γ is called a Wagner connection if it is characterized by four axioms (F2), (F3), (F6) and (F9) [2]. Therefore an *AMR*Γ is a special Wagner connection satisfying another axioms (F5).

Now we consider the following axiom :

$(F9)_1$ The *h*-connection of Γ is semi-symmetric in the *BM*Γ-sense, that is,

(2. 16) $\qquad \tau_j{}^i{}_k = \delta^i{}_j s_k - \delta^i{}_k s_j , \ s_o = -L^2 C^i{}_{1i} / (n-1)$,

where $C^i = C_j{}^i{}_k g^{jk}$.

We shall call an $M\Gamma$ the *Barthel-Matsumoto connection* if it is characterized by five axioms (F2), (F3), (F5), (F6) and $(F9)_1$, and denote it by $BM\Gamma$. Then it follows from (2.16) and Theorem 2.4 that $BM\Gamma$ is a special $AMR\Gamma$ with a scalar $f(x, y)$ defined by

$$f(x, y) = LC^i{}_{|i} / (n-1) . \tag{2.17}$$

Note 2.4. The $BM\Gamma$ has been introduced by M. Matsumoto [6]. He investigated the Barthel connection [1] and revised it to the present $BM\Gamma$. A minimal hypersurface of M can be defined under this connection.

We shall say that a scalar $f(x, y)$ on M is *RN-vanishing* if it vanishes when M becomes an $\overset{r}{M}$. The scalar $f(x, y)$ defined by (2.17) is evidently RN-vanishing.

We shall say that an $M\Gamma$ is *RN-reducible* if it reduces to $RN\Gamma$ when M becomes an $\overset{r}{M}$.

Note 2.5. If any scalar $f(x, y)$ in TMA-connections is RN-vanishing, then any TMA-connection defined by (2.7) is RN-reducible. The $BN\Gamma$ is such a good example.

§3. *TMD*-connections and Miron connections. In this section, we shall be conserned with connections whose deflexion tensors do not vanish.

We shall call an $M\Gamma$ a TMD (resp. $TMD(0)$)-connection if it is characterized by four axioms (F1), (F3) (resp. $(F3)_1$), (F4) and (F5), and denote it $TMD\Gamma$ (resp. $TMD\Gamma(0)$). From Lemmas 1.2, 1.3, 1.5 and 1.6 we can state

Theorem 3.1. *A $TMD\Gamma$ (resp. $TMD\Gamma(0)$) is uniquely determined if three tensors $T^i{}_k$, $Q_j{}^i{}_k$ and $\widetilde{C}_j{}^i{}_k$ are given as follows :*

1) $\widetilde{C}_j{}^i{}_k = C_j{}^i{}_k$ *(resp. 0)*.

2) $T^i{}_k$ *and* $Q_j{}^i{}_k$ *are a (1)p-homogeneous (1,1)-tensor and a (0)p-homogeneous (1, 2)-tensor respectively and they satisfy*

$$T^o{}_k = 0, \quad T^i{}_o + D^i{}_o = 0, \quad Q_{j\,o\,k} + D_{j\,k} = 0, \tag{3.1}$$

where $Q_{oj\,k} = D_{j\,k} = g_{j\,r} D^r{}_k$.

We shall find some $TMD\Gamma$ (or $TMD\Gamma(0)$) satisfying desirable axioms (F6) and (F7). First we can state

Lemma 3.1. *If an $M\Gamma$ satisfies axioms* (F6) *and* (F7), *then its h-connection* *is expressible in*

(3. 2) $\Gamma_j{}^i{}_k = \Gamma *_j{}^i{}_k + C_{jkr}\, T^{ri} - C_j{}^i{}_r\, T^r{}_k - C^i_{kr}\, T^r{}_j\,,$

where $T^{ri} = g^{ih}\, T^r{}_h$. *In this case, the* $M\Gamma$ *satisfies also* (F5).

Proof. From (1.5) and (F7) we first have

(3. 3) $T^j_{i\,k} + Q^j_{i\,k} = T^j_{k\,i} + Q^j_{k\,i}\,,\ T_{ijk} + Q_{ijk} = T_{kji} + Q_{kji}\,.$

Next we have (1.13) because of (F6) and Lemma 1.4. If we exchange indices i and k in (1.13), then we obtain

(3. 4) $T_{kji} + T_{jki} + Q_{kji} + Q_{jki} + 2(C_{kjr}\, T^r{}_i + P_{kji}) = 0\,.$

Substracting (3.4) from (1.13) and use (3.3), we have

$$T_{jik} - T_{jki} + Q_{jik} - Q_{jki} + 2(C_{ijr}\, T^r{}_k - C_{kjr}\, T^r{}_i) = 0\,,$$

which implies

(3. 5) $T_{jik} + Q_{jik} + 2C_{jir}\, T^r{}_k = T_{jki} + Q_{jki} + 2C_{jkr}\, T^r{}_i\,.$

Therefore it follows from (3.5) that a tensor $(T_{jik} + Q_{jik} + 2C_{jir}\, T^r{}_k)$ is symmetric in indices i and k. Applying this fact to (1.13), we have

(3. 6) $T_{ijk} + Q_{ijk} + 2P_{ijk} + T_{jki} + Q_{jki} + 2C_{jkr}\, T^r{}_i = 0\,.$

On the other hand, because of symmetry of (3.3) and (3.5) we obtain

(3. 7) $T_{jki} + Q_{jki} = T_{ikj} + Q_{ikj} = T_{ijk} + Q_{ijk} + 2C_{ijr}\, T^r{}_k - 2C_{ikr}\, T^r{}_j\,.$

Applying (3.7) to (3.6) and dividing the result by 2, we have

$$T_{ijk} + Q_{ijk} = C_{ikr}\, T^r{}_j - C_{ijr}\, T^r{}_k - C_{jkr}\, T^r{}_i - P_{ijk}\,,$$

which implies

(3. 8) $T_j{}^i{}_k + Q_j{}^i{}_k = C_{jkr}\, T^{ri} - C_j{}^i{}_r\, T^r{}_k - C^i_{kr}\, T^r{}_j - P_j{}^i{}_k\,.$

If we substitute (3.8) into (1.5), then we obtain (3.2). Further if we contract (3.8) by $y^j y^k$, then we have $T^i{}_o + Q^i_{o\,o} = 0$. Therefore this $M\Gamma$ satisfies (F5) because of Lemma 1.3. Q. E. D.

Note 3.1. Axioms (F3) (or $(F3)_1$) and (F6) always imply (F4). Axioms (F6) and (F7) do not, in general, imply (F1) without (F2).

We shall call an $M\Gamma$ a standard *TMD* (resp. *TMD* (*0*))-connection or simply an *STD* (resp. *STD* (*0*))-connection if it is characterized by four axioms (F1), (F3) (resp. $(F3)_1$)), (F6) and (F7), and denote it by $STD\Gamma$ (resp. $STD\Gamma(0)$). This connection satisfies slso (F4) and (F5) because of Lemma 3.1 and Note 3.1.

Contracting (3.8) by y^j, we have

$$D^i{}_k = -T^i{}_k - C_k{}^i{}_r\, T^r{}_o , \tag{3.9}$$

contraction of which by y^k yields $D^i{}_o = -T^i{}_o$. Therefore from (3.9) we obtain

$$T^i{}_k = C_k{}^i{}_r\, D^r{}_o - D^i{}_k,\quad T^o{}_k = -D^o{}_k . \tag{3.10}$$

Consequently from Lemma 3.1 and (3.10) we can state

Theorem 3.2. *An $STD\Gamma$ (resp. $STD\Gamma(0)$) is uniquely determined if two tensors $T^i{}_k$ (or $D^i{}_k$) and $\widetilde{C}_j{}^i{}_k$ are given as follows:*

1) $\widetilde{C}_j{}^i{}_k = C_j{}^i{}_k$ (*resp.* 0).

2) $T^i{}_k$ (*or* $D^i{}_k$) *is a (1)p-homogeneous* (1.1)*-tensor satisfying* $T^o{}_k = 0$ (*or* $D^o{}_k = 0$).

Note 3.2. When the deflexion tensor $D^i{}_k$ first given, the non-linear connection $\Gamma^i{}_k$ is given by

$$\Gamma^i{}_k = G^i{}_k + C_k{}^i{}_r\, D^r{}_o - D^i{}_k ,$$

and then *h*-connection is obtained by substitution of (3.10) into (3.2).

We shall call an $STD\Gamma$ (resp. $STD\Gamma(0)$) an *AMD* (resp. *AMD* (*0*))-connection if the tensor $T^i{}_k$ is given by (2.4) (i.e. $T^i{}_k = fLh^i{}_k$), and denote it by $AMD\Gamma$ (resp. $AMD\Gamma(0)$). In this case, we have

$$\Gamma^i{}_k = G^i{}_k + fLh^i{}_k,\quad D^i{}_k = -fLh^i{}_k , \tag{3 11}$$

$$\Gamma_j{}^i{}_k = \Gamma^*{}_j{}^i{}_k - fLC_j{}^i{}_k . \tag{3.12}$$

From (1.5), (2.5) and (3.12) we have

$$Q_j{}^i{}_k = f(h^i{}_j\, l_k + l^i\, h_{jk} - l_j\, h^i{}_k) - Lf_{\shortmid j}\, h^i{}_k - P_j{}^i{}_k - fLC_j{}^i{}_k . \tag{3.13}$$

Note 3.3. If a scalar $f(x, y)$ is *RN*-vanishing, then an $AMD\Gamma$ (or $AMD\Gamma$

(O)) is RN-reducible. In this case, the $AMD\Gamma$ (or $AMD\Gamma(O)$) is closely similar to $C\Gamma$ (or $R\Gamma$).

We shall call an $STD\Gamma$ (resp. $STD\Gamma(O)$) a C_1D(resp. R_1D)-connection if the tensor $T^i{}_k$ is given by

$$T^i{}_k = f(x, y) L C^i y_k, \tag{3.14}$$

and denote it by $C_1 D\Gamma$ (resp. $R_1 D\Gamma$). As before we obtain

$$\Gamma^i{}_k = G^i{}_k + fLC^i y_k, \quad D^i{}_k = -fL(C^i y_k + L^2 C^i_{k\,r} C^r), \tag{3.15}$$

$$\Gamma_j{}^i{}_k = \Gamma^*{}_j{}^i{}_k + fL(C_{j\,k\,r} C^r y^i - C_j{}^i{}_r C^r y_k - C^i_{k\,r} C^r y_j). \tag{3.16}$$

Differentiating (3.14) by y^j, we have

$$T_j{}^i{}_k = f_{\shortparallel j} LC^i y_k + f(l_j C^i y_k + LC^i{}_{\shortparallel j} y_k + LC^i g_{j\,k}). \tag{3.17}$$

As before we have

$$\begin{aligned} Q_j{}^i{}_k = {} & fL(C_j{}^i{}_r C^r y_k + C^i_{k\,r} C^r y_j + C_{j\,k\,r} C^r y^i - C^i{}_{\shortparallel j} y_k - C^i g_{j\,k}) \\ & - (Lf_{\shortparallel j} + fl_j) C^i y_k - P_j{}^i{}_k . \end{aligned} \tag{3.18}$$

Note 3.4. A $C_1 D\Gamma$ (or $R_1 D\Gamma$) is RN-reducible independently of $f(x, y)$. In this case, it may be considered that the $C_1 D\Gamma$ (or $R_1 D\Gamma$) is closely similar to $C\Gamma$ (or $R\Gamma$).

We shall call an $STD\Gamma$ (resp. $STD\Gamma(O)$) a $C_2 D$ (resp. $R_2 D$)-connection if the tensor $T^i{}_k$ is given by

$$T^i{}_k = f(x, y) L^3 C^i C_k, \tag{3.19}$$

and denote it by $C_2 D\Gamma$ (resp. $R_2 D\Gamma$). Then we obtain

$$\Gamma^i{}_k = G^i{}_k + fL^3 C^i C_k, \quad D^i{}_k = -fL^3 C^i C_k, \tag{3.20}$$

$$\Gamma_j{}^i{}_k = \Gamma^*{}_j{}^i{}_k + fL^3(C_{j\,k\,r} C^r C^i - C_j{}^i{}_r C^r C_k - C^i_{k\,r} C^r C_j), \tag{3.21}$$

$$T_j{}^i{}_k = (L^3 f_{\shortparallel j} + 3L^2 l_j) C^i C_k + fL^3(C^i{}_{\shortparallel j} C_k + C^i C_{k\shortparallel j}), \tag{3.22}$$

$$Q_j{}^i{}_k = fL^3(C_{j\,k\,r} C^r C^i - C_j{}^i{}_r C^r C_k - C^i_{k\,r} C^r C_j - C^i{}_{\shortparallel j} C_k - C^i C_{k\shortparallel j}) \tag{3.23}$$

$$-(L^3 f_{ij} + 3L^2 l_j)C^i C_k - P_j{}^i{}_k.$$

Note 3 . 5 . A $C_2 D\Gamma$ (or $R_2 D\Gamma$) is also RN-reducible independently of $f(x, y)$. Also in this case, the $C_2 D\Gamma$ (or $R_2 D\Gamma$) is closely similar to $C\Gamma$ (or $R\Gamma$).

Now we consider the following axiom:

(F10) The non-linear connection $\Gamma^i{}_k$ is given by $\Gamma^i{}_k = G^i{}_k$.

An $M\Gamma$ is called a Miron connection [2] if it is characterized by four axioms $(F3)_2$, (F6), (F9) and (F10). With respect to this connection, the deflexion tensor, the h-connection and the v-connection are as follows:

(3. 24) $\qquad \Gamma^i{}_k = G^i{}_k,\ D^i{}_k = s_o\, \delta^i{}_k - s^i y_k,$

(3. 25) $\qquad \Gamma_j{}^i{}_k = \Gamma^*{}_j{}^i{}_k + s_j\, \delta^i{}_k - s^i g_{jk},$

(3. 26) $\qquad \widetilde{C}_j{}^i{}_k = C_j{}^i{}_k + t_j\, \delta^i{}_k - t^i g_{jk}.$

If a Miron connection further satisfies another axiom (F5), then expressions (3.24) and (3.25) are written in

(3. 27) $\qquad \Gamma^i{}_k = G^i{}_k,\ D^i{}_k = f(x, y) L h^i{}_k,$

(3. 28) $\qquad \Gamma_j{}^i{}_k = \Gamma^*{}_j{}^i{}_k + f(l_j\, \delta^i{}_k - l^i g_{jk}).$

From (3.27) and (3.28) we have

(3. 29) $\qquad Q_j{}^i{}_k = f(l_j\, \delta^i{}_k - l^i g_{jk}) - P_j{}^i{}_k.$

We shall call a Miron connection an $MD(t)$-connection if the h-connection is given by (3.28), and denote it by $MD\Gamma(t)$. We shall, in particular, call an $MD\Gamma(t)$ an MD (resp. $MD(O)$)-connection if the v-connection is given by $\widetilde{C}_j{}^i{}_k = C_j{}^i{}_k$ (resp. $\widetilde{C}_j{}^i{}_k = 0$), and denote it by $MD\Gamma$ (resp. $MD\Gamma(O)$). Therefore an $MD\Gamma$ (resp. $MD\Gamma(O)$) is characterized by five axioms (F3) (resp. $(F3)_1$), (F5), (F6), (F9) and (F10).

Note 3 . 6 . If a scalar $f(x, y)$ is RN-vanishing, then an $MD\Gamma$ (or $MD\Gamma(O)$) is RN-reducible and closely similar to an $AMR\Gamma$ (or $AMR\Gamma(O)$).

We shall call an $M\Gamma$ a $\widetilde{H}D$ (resp. $\widetilde{B}D$)-connection if it is characterized by five axioms (F3), (resp. $(F3)_1$), (F4), (F5), (F7) and (F10), and denote it by $\widetilde{H}D\Gamma$ (resp. $\widetilde{B}D\Gamma$). From (3.1), (F7) and (F10) we have

(3. 30) $\quad Q_j{}^i{}_k = Q_k{}^i{}_j,\ \ Q_o{}^i{}_o = 0,\ \ Q_{jok} + Q_{ojk} = 0.$

Therefore a $\widetilde{HD\Gamma}$ (or $\widetilde{BD\Gamma}$) is uniquely determined if we have a tensot $Q_j{}^i{}_k$ satisfying (3.30). Such tensors are infinitely found. We shall choose three simpler ones. These are as follows :

(3. 31) $\quad Q_j{}^i{}_k = f(x, y)L^2(l_j\ C^i C_k + l_k + l_k C^i C_j - l^i C_j C_k),$

(3. 32) $\quad Q_j{}^i{}_k = f(x, y)(l_j\ h^i{}_k + l_k h^i{}_j - l^i h_{jk}).$

(3. 33) $\quad Q_j{}^i{}_k = f(x, y)(l_j\ h^i{}_k + l_k h^i{}_j - l^i h_{jk}) - P_j{}^i{}_k.$

In case of (3.31), we have

(3. 34) $\quad \Gamma^i{}_k = G^i{}_k,\ D^i{}_k = fL^3 C^i C_k,$

(3. 35) $\quad \Gamma_j{}^i{}_k = G_j{}^i{}_k + fL^2(l_j\ C^i C_k + l_k C^i C_j - l^i C_j C_k).$

We shall call a $\widetilde{HD\Gamma}$ (resp. $\widetilde{BD\Gamma}$) a *HD* (resp. *BD*)-connection if the h-connection is given by (3.35), and denote it by $HD\Gamma$ (resp. $BD\Gamma$).

Note 3 . 7 . A $HD\Gamma$ (or $BD\Gamma$) is *RN*-reducible independently of $f(x, y)$. In this case, we would like to say that the $HD\Gamma$ (or $BD\Gamma$) is closely similar to $H\Gamma$ (or $B\Gamma$).

In case of (3.32) and (3.33), we commonly obtain

(3. 36) $\quad \Gamma^i{}_k = G^i{}_k,\ \ D^i{}_k = fLh^i{}_k,$

and the h-connections are respectively given by 2) and 3) in (2.7).

We shall call a $\widetilde{HD\Gamma}$ (resp. $\widetilde{BD\Gamma}$) an *AMBD* (resp. *AMBD*(*0*))-connection or an *AMCD* (resp. *AMCD*(*0*))-connection according to whether the h-connection is given by 2) in (2.7) or by 3) in (2.7), and denote it by $AMBD\Gamma$ (resp. $AMBD\Gamma(0)$) or by $AMCD\Gamma$ (resp. $AMCD\Gamma(0)$).

Note 3 . 8 . If a scalar $f(x, y)$ is *RN*-vanishing, then any of the above connections in *RN*-reducible. In this case, an $AMBD\Gamma$ (resp. $AMCD\Gamma$) is closely similar to an $AMB\Gamma$ (resp. $AMC\Gamma$) and so is the corresponding (*0*)-connection.

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