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On Lie Derivatives on the Indicatrix Bundle over a Finsler Space

Hiroshi YASUDA

Introduction. In the papar [7][1], transformations of the tangent bundle $T(M)$ over a Finsler space M, which are generated by a general vector field on $T(M)$, were introduced, and point transformations of M and mappings among the tangent spaces were studied from a wider standpoint. The indicatrix bundle $I(M)$ over M is a hypersurface of $T(M)$ and transformations of $I(M)$ may be considered as the restrictions of those of $T(M)$ to $I(M)$ under a certain condition. In the present paper, we shall find such a condition and introduce transformations of $I(M)$, which are generated by a vector field X tangential to $I(M)$. Further we shall show that any tensor field T on $I(M)$ is obtained as the projection of a tensor field $\tilde{T}$ on $T(M)$ onto $I(M)$ and the Lie derivative of T with respect to X may be also given by the projection of that of $\tilde{T}$ onto $I(M)$. As its applications, we give conditions fox X to be a killing (or a conformal killing) vector field field with respect to a metric on $I(M)$. Finally we shall see that the D-connection on $I(M)$ is obtained as the projection of the $\tilde{D}$-connection on $T(M)$ onto $I(M)$. The terminologies and notations are referred to the paper [7] unless otherwise stated.

§1. Transformations of the indicatrix bundle. Let M be an n-dimensional Finsler space with a fundamental function $L(x, y)$ and the Cartan connection $C\Gamma=(\Gamma^{*}{}^{i}_{jk}, N^{i}_{j}, C^{i}_{jk})$, and $T(M)=\bigcup_{x\in M} T_x$, T_x being the tangent space at a point $x\in M$, be the tangent bundle over M. Then a vector field $\tilde{X}$ on $T(M)$ is given by

$$
\begin{aligned}
(1.\ 1)\qquad \tilde{X} &= u^{i}(x, y)\tilde{e}_{i} + v^{i}(x, y)\tilde{e}_{(i)} \qquad (i, j=1, 2, \dots, n)\\
&= u^{i}\partial/\partial x^{i} + (v^{i}-N^{i}_{j}u^{j})\partial/\partial y^{i}\ ,
\end{aligned}
$$

1) Numbers in brackets refer to the references at the end of the paper.

where $(\tilde{e}_i)$ and $(\tilde{e}_{(i)})$ are the horizontal and vertical bases, and u^i and v^i are the components of two vector fields on M which are homogeneous functions of degree 0 and 1 in y^i respectively.

Let $\Phi = \{\phi_t | t \in I_\varepsilon\}$ (I_ε: an open interval) be a local one-parameter group of transformations generated by $\tilde{X}$. Then each ϕ_t is defined as a solution

(1. 2) ϕ_t: $\bar{x}^i = \bar{x}^i(t; (x, y))$, $\bar{y}^i = y^i(t; (x, y))$

of the following differential equation:

(1. 3) $dx^i/dt = u^i(x, y)$, $dy^i/dt = v^i(x, y) - N^i_j(x, y)u^j(x, y)$

satisfying an initial condition

(1. 4) $\bar{x}^i(0; (x, y)) = x^i$, $\bar{y}^i(0; (x, y)) = y^i$,

Let $I(M) = \bigcup_{x \in M} I_x$, I_x being the indicatrix at a point $x \in M$, be the indicatrix bundle over M. Then $I(M)$ is a hypersurface of $T(M)$, whose local equation is given by

(1. 5) $L(x, y) = 1$.

If we denote the restriction of ϕ_t to $I(M)$ by ψ_t, namely $\psi_t = \phi_t | I(M)$, then we can state

Proposition 1. *Each restricton ψ_t is a transformation of $I(M)$ into $I(M)$ if and only if the vector field $v^i \partial/\partial x^i$ is indicatric, namely $v^i y_i = 0$,.*

Proof. From (1.3) it follows that $dL(x, y)/dt = v^i y_i/L$. Therefore, the epuation $L(\bar{x}, \bar{y}) = 1$ holds along the solution (1.2) satisfying an initial condition $L(x, y) = 1$ for $t = 0$ if and only if the vector v^i is indicatric. Q.E.D.

In this case, if we consider dt as an infinitesimal constant, then the corresponding infinitesimal transformation may be written in the form

(1. 6) $\psi_{dt}: \bar{x}^i = x^i + u^i(x, y)dt$, $\bar{y}^i = y^i + (v^i(x, y) - N^i_j u^j(x, y))dt$,

provided that the vector v^i is indicatric and $L(x, y) = 1$.

§ 2. Tensor fields on $I(M)$. We choose n vector fields ζ^i_a ($a = 1, 2, \ldots, n$) on M satisfying

(2. 1) $\zeta^i_n = l^i = y^i/L$, $g_{ij}\zeta^i_a\zeta^j_b = \delta_{ab}$,

where $g_{ij} = \frac{1}{2}\partial^2 L^2(x, y)/\partial y^i \partial y^j$. Further, if we denote the inverse of the matrix (ζ^i_a) by (ζ^a_i), then from (2.1) we have

$$g^{ij} = \sum_a \zeta^i_a \zeta^j_b, \qquad g_{ij} = \sum_a \zeta^a_i \zeta^a_i, \qquad l_i = \partial L/\partial y^i, \tag{2.2}$$
$$\zeta^i_a = g^{ij}\zeta^a_j, \qquad \zeta^i_a l_i = \zeta^a_i l^i = 0 \quad (a = 1, 2, \ldots, n-1),$$

where g^{ij} is the reciprocal tensor of g_{ij}. The coframe $(\tilde{\omega}^i, \tilde{\omega}^{(i)})$ dual to $(\tilde{e}_i, \tilde{e}_{(i)})$ on $T(M)$ is given by

$$\tilde{\omega}^i = dx^i, \qquad \tilde{\omega}^{(i)} = dy^i + N^i_j dx^i. \tag{2.3}$$

Now we take an adapted orthogonal frame $(e_a, e_{(a)})$ and its coframe $(\omega^a, \omega^{(a)})$ on $I(M)$. These are defined as follows [6]:

$$e_a = \zeta^i_a e_i, \qquad e_{(a)} = \zeta^i_a e_{(i)}, \qquad \omega^a = \zeta^a_i \omega^i, \qquad \omega^{(a)} = \zeta^a_i \omega^{(i)}, \tag{2.4}$$

provided $L(x, y) = 1$ and

$$e_i = \tilde{e}_i, \qquad e_{(i)} = h^j_i \tilde{e}_{(j)}, \qquad \omega^i = \tilde{\omega}^i, \qquad \omega^{(i)} = h^i_j \tilde{\omega}^{(j)}, \qquad \textit{where} \tag{2.5}$$
$$h^i_j = h^i_j - l^i l_j.$$

Next we can take another frame and its coframe on $I(M)$ given by

$$(e_i, e_{(a)}), \qquad (\omega^i, \omega^{(a)}). \tag{2.6}$$

Then we can state

Proposition 2. *The frame and coframe in (2.6) are the projections of $(\tilde{e}_i, \tilde{e}_{(i)})$ and $(\tilde{\omega}^i, \tilde{\omega}^{(i)})$ on $T(M)$ onto $I(M)$.*

Proof. At any point (x, y) of $I(M)$ we have (2.5). $(\tilde{e}_{(i)})$ and $(\tilde{\omega}^{(i)})$ are the frame and coframe on the tangent space T_x. On the other hand, the projection of a tensor, for example, T^i_j on T_x onto the indicatrix I_x is given by $T^a_\beta = T^i_j \zeta^a_i \zeta^j_\beta$ [4]. Then from (2.2), (2.3) and (2.5) we have

$$T^a_\beta e_{(a)} \otimes \omega^{(\beta)} = T^k_h h^i_k h^h_j e_{(i)} \otimes \omega^{(j)} = T^k_h h^i_k h^h_j \tilde{e}_{(i)} \otimes \tilde{\omega}^{(j)}. \qquad \text{Q.E.D.}$$

If the vector v^i is indicatric, then the projection X of the vector field $\tilde{X}$ in (1.1) onto $I(M)$ is given by

$$X = u^i e_i + v^a e_{(a)} = u^i e_i + v^i \tilde{e}_{(i)} = u^i \partial/\partial x^i + (v^i - N^i_j u^j)\partial/\partial y^i, \tag{2.7}$$

where $v^a = v^i \zeta^a_i$ and $L(x, y) = 1$. Therefore, on $I(M)$ $\tilde{X}$ is identified with X and the infinitesimal transformation generated by X corresponds to (1.6).

A tensor field $\widetilde{T}$ of (1, 1)-type on $T(M)$ is given by

$$
\begin{aligned}
\widetilde{T} &= T^{Y}_{Z}(x, y)\tilde{e}_{Y} \otimes \tilde{\omega}^{Z} \qquad (Y, Z = 1, 2, \dots, 2n) \\
&= T^{i}_{j}\tilde{e}_{i} \otimes \tilde{\omega}^{j} + T^{(i)}_{j}\tilde{e}_{(i)} \otimes \tilde{\omega}^{j} + T^{i}_{(j)}\tilde{e}_{i} \otimes \tilde{\omega}^{(j)} + T^{\{i\}}_{\{j\}}\tilde{e}_{(i)} \otimes \tilde{\omega}^{(j)},
\end{aligned} \tag{2. 8}
$$

where T^{i}_{j}, $T^{(i)}_{j}$, $T^{i}_{(j)}$ and $T^{\{i\}}_{\{j\}}$ are the components of four tensor fields on M.

On the other hand, a tensor field T on $I(M)$ is given by

$$
\begin{aligned}
T &= T^{A}_{B}e_{A} \otimes \omega^{B} \qquad (A, B = 1, 2, \dots, 2n-1) \\
&= T^{a}_{b}e_{a} \otimes \omega^{b} + T^{(\alpha)}_{b}e_{(\alpha)} \otimes \omega_{b} + T^{a}_{(\beta)}e_{a} \otimes \omega^{(\beta)} + T^{\{\alpha\}}_{\{\beta\}}e_{(\alpha)} \otimes \omega^{(\beta)},
\end{aligned} \tag{2. 9}
$$

where

$$
\begin{aligned}
&T^{a}_{b} = T^{i}_{j}\zeta^{a}_{i}\zeta^{j}_{b}, \qquad T^{(\alpha)}_{b} = T^{(i)}_{j}\zeta^{\alpha}_{i}\zeta^{j}_{b}, \qquad T^{\alpha}_{(\beta)} = T^{i}_{(j)}\zeta^{a}_{i}\zeta^{j}_{\beta}, \\
&T^{\{\alpha\}}_{\{\beta\}} = T^{\{i\}}_{\{j\}}\zeta^{a}_{i}\zeta^{j}_{\beta}.
\end{aligned} \tag{2. 10}
$$

Further it is seen from (2.2), (2.4), (2.5) and (2.10) that (2.9) is expressed in the form

$$
\begin{aligned}
T &= T^{i}_{j}e_{i} \otimes \omega^{j} + T^{(\alpha)}_{j}e_{(\alpha)} \otimes \omega^{j} + T^{i}_{(\beta)}e_{i} \otimes \omega^{(\beta)} + T^{\{\alpha\}}_{\{\beta\}}e_{(\alpha)} \otimes \omega^{(\beta)} \\
&= T^{i}_{j}e_{i} \otimes \omega^{j} + {}'T^{(i)}_{j}e_{(i)} \otimes \omega^{j} + {}'T^{i}_{(j)}e_{i} \otimes \omega^{(j)} + {}'T^{\{i\}}_{\{j\}}e_{(i)} \otimes \omega^{(j)},
\end{aligned} \tag{2. 11}
$$

where $T^{(\alpha)}_{j} = T^{(i)}_{j}\zeta^{\alpha}_{i}$, $T_{(\beta)} = T^{i}_{(j)}\zeta^{i}_{\beta}$ and the symbol [′] indicates the indicatrization with respect to the vertical indices (i), namely

$$
{}'T^{(i)}_{j} = T^{(r)}_{j}h^{i}_{r}, \qquad {}'T^{i}_{(j)} = T^{i}_{(r)}h^{r}_{j}, \qquad {}'T^{\{i\}}_{\{j\}} = T^{\{r\}}_{\{s\}}h^{i}_{r}h^{s}_{j}. \tag{2. 12}
$$

Thus from (2.8), (2.11) and Proposition 2 we can state

Proposition 3. *Any tensor field T on $I(M)$ is the projection of a tensor field $\widetilde{T}$ on $T(M)$ onto $I(M)$ and the field T is obtained from $\widetilde{T}$ in the way as (2.11).*

§ 3. Lie dervatives. Let $\widetilde{X}$ be a vector field on $T(M)$ defined by (1.1). Then the Lie derivative $L_{\tilde{X}}\widetilde{T}$ of tensor field $\widetilde{T}$ in (2.8) with respect to $\widetilde{X}$ is given as follows [7]:

$$
\begin{aligned}
L_{\tilde{X}}\widetilde{T} &= L_{\tilde{X}}T^{i}_{j}\tilde{e}_{i} \otimes \tilde{\omega}^{j} + L_{\tilde{X}}T^{(i)}_{j}\tilde{e}_{(i)} \otimes \tilde{\omega}^{j} + L_{\tilde{X}}T^{i}_{(j)}\tilde{e}_{i} \otimes \tilde{\omega}^{(j)} \\
&\quad + L_{\tilde{X}}T^{\{i\}}_{\{j\}}\tilde{e}_{(i)} \otimes \tilde{\omega}^{(j)}, \\
L_{\tilde{X}}T^{i}_{j}\tilde{e}_{i} \otimes \tilde{\omega}^{j} &= (L_{\tilde{X}}T^{i}_{j})\tilde{e}_{i} \otimes \tilde{\omega}^{j} + (T^{i}_{r}u^{r}_{\|j})\tilde{e}_{i} \otimes \tilde{\omega}^{(j)} \\
&\quad - (T^{r}_{j}W^{i}_{r})\tilde{e}_{(i)} \otimes \tilde{\omega}_{j},
\end{aligned} \tag{3. 1}
$$

$$
(3.\ 1)\qquad
\begin{aligned}
L_{\tilde X}\tilde T^{(i)}_{j}\tilde e_{(i)}\otimes\tilde\omega^{j} &= (L_{\tilde X}T^{(i)}_{j})\tilde e_{(i)}\otimes\tilde\omega^{j} - (T^{(r)}_{j}u^{i}{}_{\|r})\tilde e_{i}\otimes\tilde\omega^{(j)}\\
&\quad + (T^{(i)}_{r}u^{r}{}_{\|j})\tilde e_{(i)}\otimes\tilde\omega^{(j)},\\
L_{\tilde X}T^{\{i\}}_{\{j\}}\tilde e_{i}\otimes\tilde\omega^{(j)} &= (L_{\tilde X}T^{i}_{(j)})\tilde e_{i}\otimes\tilde\omega^{(j)} + (T^{i}_{(r)}W^{r}_{j})\tilde e_{i}\otimes\tilde\omega^{j}\\
&\quad - (T^{r}_{(j)}W^{i}_{r})\tilde e_{(i)}\otimes\tilde\omega^{(j)},\\
L_{\tilde X}T^{\{i\}}_{\{j\}}\tilde e_{(i)}\otimes\tilde\omega^{(j)} &= (L_{\tilde X}T^{\{i\}}_{\{j\}}\tilde e_{(i)})\otimes\tilde\omega^{(j)} - (T^{(r)}_{j}u^{i}{}_{\|r})\tilde e_{i}\otimes\tilde\omega^{(j)}\\
&\quad + (T^{\{i\}}_{\{r\}}W^{r}_{j})\tilde e_{(i)}\otimes\tilde\omega^{j},
\end{aligned}
$$

where

$$
(3.\ 2)\qquad
\begin{aligned}
&L_{\tilde X}T^{i}_{j} = T^{i}_{j|r}u^{r} - T^{r}_{j}u^{i}{}_{|r} + T^{i}_{r}u^{r}{}_{|j} + T^{i}_{j\|r}v^{r},\\
&L_{\tilde X}T^{(i)}_{r} = T^{(i)}_{j}{}_{|r}u^{r} - T^{(r)}_{j}V^{i}_{r} + T^{(i)}_{r}u^{r}{}_{|j} + T^{(i)}_{j}{}_{\|r}v^{r},\\
&L_{\tilde X}T^{i}_{(j)} = T^{i}_{(j)|r}u^{r} - T^{r}_{(j)}u^{i}{}_{|r} + T^{i}_{(r)}V^{r}_{j} + T^{i}_{(j)\|r}v^{r},\\
&L_{\tilde X}T^{\{i\}}_{\{j\}} = T^{\{i\}}_{\{j\}|r}u^{r} - T^{\{r\}}_{\{j\}}V^{i}_{r} + T^{\{i\}}_{\{r\}}V^{r}_{j} + T^{\{i\}}_{\{j\}\|r}v^{r},\\
&W^{i}_{j} = v^{i}{}_{|j} + C^{i}_{jk|0}v^{k} + K^{i}_{0jk}u^{r}, \qquad V^{i}_{j} = v^{i}{}_{\|j} - C^{i}_{jk|0}u^{k}.
\end{aligned}
$$

Let X and T be the projections of $\tilde{X}$ and $\tilde T$ onto $I(M)$. Then, if the vector v^i is indicatric, then we have seen that $X=\tilde{X}$ on $I(M)$. And the Lie derivative $L_X\,T$ of T with respect X can be regarded as the projection of (3.1) onto $I(M)$. Consequently by virtue of (3.1) and Proposition 3 we have

$$
(3.\ 3)\qquad
\begin{aligned}
L_XT &= L_XT^{i}_{j}e_{i}\otimes\omega^{j} + L_XT^{(\alpha)}_{j}e_{(\alpha)}\otimes\omega^{j} + L_XT^{i}_{(\beta)}e_{i}\otimes\omega^{(\beta)}\\
&\quad + L_XT^{\{\alpha\}}_{\{\beta\}}e_{(\alpha)}\otimes\omega^{(\beta)},\\
L_XT^{i}_{j}e_{i}\otimes\omega^{j} &= (L_XT^{i}_{j})e_{i}\otimes\omega^{j} + {}^{*}(T^{i}_{r}u^{r}{}_{\|j})e_{i}\otimes\omega^{(j)}\\
&\quad - {}^{*}(T^{r}_{j}W^{i}_{r})e_{(i)}\otimes\omega^{j},\\
L_XT^{(\alpha)}_{j}e_{(\alpha)}\otimes\omega^{j} &= {}'(L_XT^{(i)}_{j})e_{(i)}\otimes\omega^{j} - {}^{*}(T^{r}_{j}u^{i}{}_{\|r})e_{i}\otimes\omega^{(j)}\\
&\quad + {}^{*}(T^{(i)}_{r}u^{r}{}_{\|j})e_{(i)}\otimes\omega^{(j)},\\
L_XT^{i}_{(\beta)}e_{i}\otimes\omega^{(\beta)} &= {}'(L_XT^{i}_{(j)})e_{i}\otimes\omega^{(j)} + (T^{i}_{(r)}W^{r}_{j})e_{i}\otimes\omega^{j}\\
&\quad - {}^{*}(T^{r}_{(j)}W^{i}_{r})e_{(i)}\otimes\omega^{(j)},\\
L_XT^{\{\alpha\}}_{\{\beta\}}e_{(\alpha)}\otimes\omega^{(\beta)} &= {}'(L_XT^{\{i\}}_{\{j\}})e_{(i)}\otimes\omega^{(j)} - {}^{*}(T^{(r)}_{j}u^{i}{}_{\|j})e_{i}\otimes\omega^{(j)}\\
&\quad + {}^{*}(T^{\{i\}}_{\{r\}}W^{r}_{j})e_{(i)}\otimes\omega^{j},
\end{aligned}
$$

where the symbol [*] indicates the indicatrization with respect to the indices corresponding to vertical bases; for example, in the third term of the right side of the third expression in (3.3), ${}^{*}(T^{(i)}_{r}u^{r}{}_{\|j})=T^{(s)}_{r}u^{r}{}_{\|t}h^{i}_{s}h^{t}_{j}$, the indices i and j corresponding to $e_{(i)}$ and $\omega^{(j)}$. Thus we have

Proposition 4. *The Lie derivative L_XT of a tensor field T on $I(M)$*

with respect to a vector field X is given by (3.3) together with (3.2).

A metric on $I(M)$ may by given by the projection of a metric on $T(M)$ onto $I(M)$. On $T(M)$ we take the following metric:

$$\tilde{G} = \tilde{G}_1 + \tilde{G}_2, \quad \text{where } \tilde{G}_1 = g_{ij}\tilde{\omega}^i \otimes \tilde{\omega}^j \text{ and } \tilde{G}_2 = g_{ij}\tilde{\omega}^{(i)} \otimes \tilde{\omega}^{(j)}. \tag{3.4}$$

Then it follows from proposition 3 that the projection G of $\tilde{G}$ onto $I(M)$ is given by

$$G = G_1 + G_2, \quad \text{where } G_1 = g_{ij}\omega^i \otimes \omega^j \text{ and } G_2 = h_{ij}\omega^{(i)} \otimes \omega^{(j)}. \tag{3.5}$$

Noticing $u^i{}_{\|j}y^i = 0$ and $v^i y_i = 0$, from Proposition 4 we have

$$\begin{aligned} L_X G_1 = (u_{i|j} + u_{j|i} + 2C_{ijk}v^k)\omega^i \otimes \omega^j + g_{ir}u^r{}_{\|j}\omega^i \otimes \omega^{(j)} \\ + g_{rj}u^r{}_{\|i}\omega^{(i)} \otimes \omega^j, \end{aligned} \tag{3.6}$$

$$\begin{aligned} L_X G_2 = (v_i|_j + v_j|_i - 2C_{ijk|0}u^k)\omega^{(i)} \otimes \omega^{(j)} + g_{ir}W^r_s h^s_j \omega^i \otimes \omega^{(j)} \\ + g_{rj}W^r_s h^s_i \omega^{(i)} \otimes \omega^j. \end{aligned} \tag{3.7}$$

Since $l^i = y^i/L = y^i$ on $I(M)$, we have

$$\omega^{(i)} = h^i_j(dl^j + \Gamma^{*j}_{kr}l^r dx^k) = dl^i + N^i_j dx^j = Dl^i, \tag{3.8}$$

which is indicatric. Then we have

Lemma. *For a vector Z_i, $Z_i\omega^{(i)} = 0$ holds if and only if*

$$'Z_i = Z_j h^j_i = 0. \tag{3.9}$$

Proof. We have $Z_i\omega^{(i)} = Z_i(h^i_j\omega^{(j)}) = Z_i(\zeta^i_a\zeta^a_j)\omega^{(j)} = Z_i\zeta^i_a\omega^{(a)} = 0$. Sincc $\omega^{(a)}$ $(a = 1, 2, \dots, n-1)$ are independent, we get $Z_i\zeta^i_a = 0$ and hence (3.9) follows. The converse is evident. Q.E.D.

Remark. If Z_i is indicatric, then $Z_i\omega^{(i)} = 0$ implies $Z_i = 0$ because of (3.9).

Taking account of homogeneity of u^i and v^i, from (3.2), (3.6), (3.7) and Remark we can state

Theorem 1. *A vector field X on I(M) is a killing vector one with respect to G if and only if the following equations hold:*

$$\begin{aligned} u_{i|j} + u_{j|i} + 2C_{ijk}v^k = 0, \quad v_i|_j + v_j|_i - 2C_{ijk|0}u^k = 0, \\ u^i{}_{\|j} = 0, \quad (v^i{}_r + C^i_{rk|0}v^k + K^i_{0rk}u^k)h^r_j = 0. \end{aligned} \tag{3.10}$$

Immediately we have

Corollary 1.1. *A vector field $X = u^i e_i$ on $I(M)$ is killing vector one with respect G if and only if the following equations hold:*

(3. 11) $$u_{i|j} + u_{j|i} = 0, \quad u^i{}_{\|j} = 0, \quad C_{ijk|0}u^k = (K^i_{0jk} - K^i_{00k} l_j/L)u^k = 0.$$

Corollary 1.2. *A vector field $X = v^i e_{(i)}$ on $I(M)$ is a killing vector one with respect to G if and only if the following equations hold:*

(3. 12) $$C_{ijk}v^k = 0, \quad v_{i\|j} + v_{j\|i} = 0, \quad v^i{}_{|j} + C^i_{j\,k|0} v^k - v^i{}_{|0} l_j/L = 0.$$

For a conformal killing vector field, we have

Theorem 2. *A vector field X on $I(M)$ is a conformal killing vector one with respect to G if and only if the following equations hold:*

(3. 13) $$u_{i|j} + u_{j|i} + 2C_{ijk}v^k = 2\rho g_{ij}, \quad v_i|_j + v_j|_i - 2C_{ijk|0}u^k = 2\rho h_{ij},$$
$$u^i{}_{\|j} = 0, \quad (v^i{}_{|r} + C^i_{r\,k|0} v^k + K^i_{0\,rk}u^k)h^r_j = 0.$$

In this case, the scalar ρ is given by

(3. 14) $$\rho = (u^i{}_{|i} + C_i v^i)/n = (v^i{}_{|i} - C_{i|0}u^i)/(n-1).$$

Corollary 2.1. *A vector field $X = u^i e_i$ on $I(M)$ is a conformal killing vector one if and only if the following equations hold:*

(3. 15) $$u_{i|j} + u_{j|i} = \rho g_{ij}, \quad C_{ijk|0}u^k = -\rho h_{ij}, \quad u^i{}_{\|j} = 0,$$
$$(K^i_{0\,jk} - K^i_{0\,0k} l_j/L)u^k = 0.$$

In this case, the scalar is given by $\rho = u^i{}_{|i}/n = -C_{i|0}u^i/(n-1)$.

Corollary 2.2. *A vector field $X = v^i e_{(i)}$ on $I(M)$ is a conformal killing vector one if and only if X is a killing vector field.*

Proof. From (3.13) we have $C_{ijk}v^k = \rho g_{ij}$, contraction of which by $l^i l^j$ yields $\rho = 0$. Q.E.D.

§4. Connections. On $T(M)$ we consider a connection such that increments of the frame $(\tilde{e}_i, \tilde{e}_{(i)})$ are given by

(4. 1) $$d\tilde{e}_j = \tilde{\omega}^i_j \tilde{e}_i + \tilde{\omega}^{(i)}_j \tilde{e}_{(i)}, \quad d\tilde{e}_{(j)} = \tilde{\omega}^i_{(j)}\tilde{e}_i + \tilde{\omega}^{(i)}_{(j)}\tilde{e}_{(i)},$$

where

(4. 2) $$\tilde{\omega}^i_j = \Gamma^{*i}{}_{j\,k}\tilde{\omega}^k + C^i_{j\,k}\tilde{\omega}^{(k)}, \quad \tilde{\omega}^{(i)}_{(j)} = \tilde{\omega}^i_j, \quad \tilde{\omega}^{(i)}_j = -(B^i{}_{jk}\tilde{\omega}^k + P^i{}_{jk}\tilde{\omega}^{(k)}),$$

$$\tilde{\omega}^{i}_{(j)} = B_j{}^i{}_k \tilde{\omega}^k + P_j{}^i{}_k \tilde{\omega}^{(k)}, \quad B^i{}_{jk} = LC^i{}_{jk} + R_0{}^i{}_{jk}/L, \quad P^i{}_{jk} = C^i{}_{jk|0}\,,$$
$$B_j{}^i{}_k = B^r{}_{sk} g_{rj} g^{si}, \quad P_j{}^i{}_k = P^i{}_{jk}\ (\textit{necessarily valid})\,.$$

We shall call such a connection the $\tilde{D}$-*connection* on $T(M)$. Now we shall seek for the projection of this connection onto $I(M)$. First, according to Proposition 2 the relation (4.1) may be written in the form

$$(4.3)\qquad de_j = \omega^i_j e_i + \omega^{(\alpha)}_j e_{(\alpha)}\,, \qquad de_{(\beta)} = \omega^{i}_{(\beta)} e_i + \omega^{(\alpha)}_{(\beta)} e_{(\alpha)}\,.$$

Next, for ω^i_j we have

$$(4.4)_1\qquad \omega^i_j = \Gamma^*{}_j{}^i{}_k \omega^k + C_j{}^i{}_k \zeta^k_\alpha \omega^{(\alpha)} = \Gamma^*{}_j{}^i{}_k \omega^k + C_j{}^i{}_k \omega^{(k)} \quad (L(x, y) = 1)\,,$$

which are, because of (3.8), the connection forms of Cartan.

For $\omega^{(\alpha)}_j$ and $\omega^{i}_{(\beta)}$, we have

$$(4.4)_2\qquad \omega^{(\alpha)}_j = \omega^{(i)}_j \zeta^\alpha_i\,, \qquad \omega^{i}_{(\beta)} = \omega^{i}_{(j)} \zeta^j_\beta\,,$$

where $\omega^{(i)}_j = -(B^i{}_{jk}\omega^k + P^i{}_{jk}\omega^{(k)})$ and $\omega^i_{(j)} = B_j{}^i{}_k \omega^k + P_j{}^i{}_k{}^{(k)}$.

For $\omega^{(\alpha)}_{(\beta)}$, we have

$$(4.4)_3\qquad \omega^{(\alpha)}_{(\beta)} = \zeta^j_\beta \omega^i_j \zeta^\alpha_i + \zeta^\alpha_i d\zeta^i_\beta\,.$$

Thus we have obtained the projection of the $\tilde{D}$-connection onto $I(M)$. On the other hand, on $I(M)$ we have the D-connection [6]. In this case, we can prove that the above projection is identified with the D-connection. Hence we have

Proposition 5. *The D-connection on I(M) is obtained as the projection of the $\tilde{D}$-connection on T(M) onto I(M).*

On $T(M)$ we can take another connection. This is defined as follows: In (4.1) and (4.2), $\tilde{\omega}^{(i)}_j = \tilde{\omega}^i_{(j)} = 0$, namely

$$(4.5)\qquad d\tilde{e}_j = \tilde{\omega}^i_j \tilde{e}_i\,, \qquad d\tilde{e}_{(j)} = \tilde{\omega}^{(i)}_{(j)} \tilde{e}_{(i)}\,.$$

Such a connection will be called the $\tilde{K}$-*connection* on $T(M)$. On the other hand, on $I(M)$ we have the K-connection [5], [6]. Concerning this, we have

Corollary 5.1. *The K-connection on I(M) is obtained as the projection of the $\tilde{K}$-connection on T(M) onto I(M).*

As for the Lie derivatives of the above connections, we shall discuss them in later papers.

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(Mathematics, Asahikawa Medical College)