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On infinitesimal conformal transformations of the tangent bundles with the metric II+III over Riemannian manifolds

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Introduction.

Let M be an n-dimensional Riemannian manifold with a metric g and let V be a vector field on M. Let us consider the local one-parameter group $\{\phi_t\}$ of local transformations of M generated by V. Then V is called an infinitesimal conformal transformation, if each ϕ_t is a local conformal transformation of M. It is well known that V is an infinitesimal conformal transformation if and only if there exists a scalar function ρ on M such that $\mathcal{L}_V g = 2\rho g$, where $\mathcal{L}_V$ denotes the Lie derivation with respect to the vector field V, especially V is called an infinitesimal homothetic one when ρ is constant.

Let $T(M)$ be the tangent bundle over M, and let Φ be a transformation of $T(M)$. Then Φ is called a fibre-preserving transformation, if it preserves the fibres. Let X be a vector field on $T(M)$, and let us consider the local one-parameter group $\{\Phi_t\}$ of local transformations of $T(M)$ generated by X. Then X is called an infinitesimal fibre-preserving transformation, if each Φ_t is a local fibre-preserving transformation of $T(M)$. Clearly an infinitesimal fibre-preserving transformation on $T(M)$ induces an infinitesimal transformation in the base space M. An infinitesimal fibre-preserving transformation X on $T(M)$ is called an infinitesimal fibre-preserving conformal transformation, if each Φ_t is a local fibre-preserving conformal transformation of $T(M)$. Let G be a Riemannian or a pseudo-Riemannian metric of $T(M)$. It is well known that X is an infinitesimal conformal transformation of $T(M)$ if and only if there exists a scalar function Ω on $T(M)$ such that $\mathcal{L}_X G = 2\Omega G$, where $\mathcal{L}_X$ denotes the Lie derivation with respect to the vector field X.

In the previous papers [**1**], [**2**], [**3**], we proved the following theorems.

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THEOREM. *Let M be an n-dimensional Riemannian manifold, and let $T(M)$ be its tangent bundle with the metric II. Then every infinitesimal fibre-preserving conformal transformation X on $T(M)$ naturally induces an infinitesimal projective transformation V on M. Furthermore the correspondence $X \to V$ gives a homomorphism of the Lie algebra of infinitesimal fibre-preserving conformal transformations on $T(M)$ onto the Lie algebra of infinitesimal projective transformations on M, and the kernel of this homomorphism is naturally isomorphic onto the Lie algebra of infinitesimal isometries of M.*

THEOREM. *Let M be an n-dimensional Riemannian manifold, and let $T(M)$ be its tangent bundle with the metric $I+III$. Then every infinitesimal fibre-preserving conformal transformation X is a homothetic one and it induces an infinitesimal homothetic transformation V on M. Furthermore the correspondence $X \to V$ gives a homomorphism of the Lie algebra of infinitesimal fibre-preserving conformal transformations on $T(M)$ onto the Lie algebra of infinitesimal homothetic transformations on M, and the kernel of this homomorphism is naturally isomorphic onto the the Lie algebra of infinitesimal isometries of M.*

THEOREM. *Let M be an n-dimensional Riemannian manifold, and let $T(M)$ be its tangent bundle with the metric $I+II$. Then every infinitesimal fibre-preserving conformal transformation X on $T(M)$ naturally induces an infinitesimal projective transformation V on M. Furthermore the correspondence $X \to V$ gives a homomorphism of the Lie algebra of infinitesimal fibre-preserving conformal transformations on $T(M)$ onto the Lie algebra of infinitesimal projective transformations on M.*

The purpose of the present paper is to prove the following theorem.

THEOREM. *Let M be an n-dimensional Riemannian manifold, and let $T(M)$ be its tangent bundle with the metric $II+III$. Then every infinitesimal fibre-preserving conformal transformation X is a homothetic one and it induces an infinitesimal homothetic transformation V on M. Furthermore the correspondence $X \to V$ gives a homomorphism of the Lie algebra of infinitesimal fibre-preserving conformal transformations on $T(M)$ onto the Lie algebra of infinitesimal homothetic transformations on M, and the kernel of this homomorphism is naturally isomorphic onto the Lie algebra of infinitesimal isometries of M.*

§1. Preliminaries.

Let Γ_{ji}^h be the coefficients of the Riemannian connection of M, then $y^a\Gamma_{ai}^h$ can be regarded as coefficients of the non-linear connection of $T(M)$, where (x^h, y^h) the induced coordinates in $T(M)$. We define

$$X_h = \frac{\partial}{\partial x^h} - y^a \Gamma^m_{ah} \frac{\partial}{\partial y^m} \quad \text{and} \quad X_{\bar{h}} = \frac{\partial}{\partial y^h},$$

then $\{X_h, X_{\bar{h}}\}$ are called the adapted frame of $T(M)$, and let $\{dx^h, \delta y^h\}$ be the dual basis of $\{X_h, X_{\bar{h}}\}$.

We can easily prove the following lemma.

LEMMA 1. *The Lie brackets satisfy the following*:

(1) $$[X_i, X_j] = y^r K_{jir}{}^m X_{\bar{m}},$$

(2) $$[X_i, X_{\bar{j}}] = \Gamma^m_{ji} X_{\bar{m}},$$

(3) $$[X_{\bar{i}}, X_{\bar{j}}] = 0,$$

where $K_{jir}{}^m$ denote the components of the curvature tensor of M.

Let X be an infinitesimal fibre-preserving transformation on $T(M)$ and $(v^h, v^{\bar{h}})$ the components of X with respect to the adapted frame $\{X_h, X_{\bar{h}}\}$. Then X is fibre-preserving if and only if v^h depend only on the variables (x^h). Clearly X induces an infinitesimal transformation V with the components v^h in the base space M. Let $\mathcal{L}_X$ be the Lie derivation with respect to X, then we have the following lemma.

LEMMA 2. (See [1]). *The Lie derivatives of the adapted frame and the dual basis are given as follows*:

(1) $\mathcal{L}_X X_h = -\partial_h v^a X_a + \{y^b v^c K_{hcb}{}^a - v^b \Gamma^a_{bh} - X_h(v^{\bar{a}})\} X_{\bar{a}},$

(2) $\mathcal{L}_X X_{\bar{h}} = \{v^b \Gamma^a_{bh} - X_{\bar{h}}(v^{\bar{a}})\} X_{\bar{a}},$

(3) $\mathcal{L}_X dx^h = \partial_m v^h dx^m,$

(4) $\mathcal{L}_X \delta y^h = -\{y^b v^c K_{mcb}{}^h - v^{\bar{b}} \Gamma^h_{bm} - X_m(v^{\bar{h}})\} dx^m - \{v^b \Gamma^h_{bm} - X_{\bar{m}}(v^{\bar{h}})\} \delta y^m.$

Let g be a Riemannian metric of M with the components g_{ji}, then we see that

I : $G = g_{ji} dx^j dx^i,$

II : $G = 2g_{ji} dx^j \delta y^i,$

III: $G = g_{ji} \delta y^j \delta y^i,$

are all quadratic differential forms defiend globally in $T(M)$ and that

II: $2g_{ji} dx^j \delta y^i,$

I+II: $g_{ji} dx^j dx^i + 2g_{ji} dx^j \delta y^i,$

I+III: $g_{ji} dx^j dx^i + g_{ji} \delta y^j \delta y^i,$

II+III: $2g_{ji} dx^j \delta y^i + g_{ji} \delta y^j \delta y^i,$

are all non-singular and consequently can be regarded as Riemannian or pseudo-Riemannian metrics in $T(M)$.

LEMMA 3. (See [1]). *The Lie derivatives $\mathcal{L}_X G_I$, $\mathcal{L}_X G_{II}$ and $\mathcal{L}_X G_{III}$ are given as follows*:

(1) $\pounds_X G_I = (\pounds_V g_{ji}) dx^j dx^i$,

(2) $\frac{1}{2}\pounds_X G_{II} = -g_{jm}\{y^b v^c K_{icb}{}^m - v^{\bar{b}}\Gamma^m_{bi} - X_i(v^{\bar{m}})\} dx^j dx^i$
$\quad + \{\pounds_V g_{ij} - g_{jm}\nabla_i v^m + g_{jm} X_{\bar{i}}(v^{\bar{m}})\} dx^j \delta y^i$,

(3) $\pounds_X G_{III} = -2g_{mi}\{y^b v^c K_{jcb}{}^m - v^{\bar{b}}\Gamma^m_{bj} - X_j(v^{\bar{m}})\} dx^j \delta y^i + 2g_{jm} X_{\bar{i}}(v^{\bar{m}})\delta y^j \delta y^i$,

where $\pounds_V g_{ji}$ denote the components of the Lie derivative $\pounds_V g$ and $\nabla_i v^m$ the components of the covariant derivative of V.

§2. Infinitesimal conformal transformations of the tangent bundles with the metric II+III.

Let $T(M)$ be the tangent bundle over M with the metric II+III, and let X be an infinitesimal fibre-preserving conformal transformation on $T(M)$, that is, there exists a scalar function Ω on $T(M)$ such that $\pounds_X G_{II+III} = 2\Omega G_{II+III}$. Then from Lemma 3, we have

$$-2g_{im}\{y^b v^c K_{jcb}{}^m - v^{\bar{b}}\Gamma^m_{bj} - X_j(v^{\bar{m}})\}dx^i dx^j + 2\{\pounds_V g_{ij} - g_{im}\nabla_j v^m + g_{im}X_{\bar{j}}(v^{\bar{m}}) - g_{mj}y^b v^c K_{icb}{}^m$$
$$+ g_{mj}v^{\bar{b}}\Gamma^m_{bi} + g_{mj}X_i(v^{\bar{m}})\}dx^i\delta y^j + 2g_{mi}X_{\bar{j}}(v^{\bar{m}})\delta y^i \delta y^j = 4\Omega g_{ij}dx^i\delta y^j + 2\Omega g_{ij}\delta y^i \delta y^j,$$

from which we get

(2.1) $g_{im}\{y^b v^c K_{jcb}{}^m - v^{\bar{b}}\Gamma^m_{bj} - X_j(v^{\bar{m}})\} + g_{jm}\{y^b v^c K_{icb}{}^m - v^{\bar{b}}\Gamma^m_{bi} - X_i(v^{\bar{m}})\} = 0$,

(2.2) $\pounds_V g_{ij} - g_{mi}\nabla_j v^m + g_{mi}X_{\bar{j}}(v^{\bar{m}}) - g_{jm}\{y^b v^c K_{icb}{}^m - v^{\bar{b}}\Gamma^m_{bi} - X_i(v^{\bar{m}})\} = 2\Omega g_{ij}$,

(2.3) $g_{mi}X_{\bar{j}}(v^{\bar{m}}) + g_{mj}X_{\bar{i}}(v^{\bar{m}}) = 2\Omega g_{ij}$.

Proposition 1. *The vector field V with the components (v^h) is an infinitesimal conformal transformation on M.*

Proof. From the equation (2.2), we have

$$2\pounds_V g_{ij} - g_{mi}\nabla_j v^m + g_{mi}X_{\bar{j}}(v^{\bar{m}}) - g_{mj}\nabla_i v^m + g_{mj}X_{\bar{i}}(v^{\bar{m}}) - g_{jm}\{y^b v^c K_{icb}{}^m - v^{\bar{b}}\Gamma^m_{bi} - X_i(v^{\bar{m}})\}$$
$$- g_{im}\{y^b v^c K_{jcb}{}^m - v^{\bar{b}}\Gamma^m_{bj} - X_j(v^{\bar{m}})\} = 4\Omega g_{ij}.$$

Substituting the equations (2.1) and (2.3) into the above equation, we get $\pounds_V g_{ij} = 2\Omega g_{ij}$. This shows the scalar function Ω on $T(M)$ depends only on the variables (x^h) with respect to the induced coordinates (x^h, y^h), thus we can regard Ω is a function on M, and V is an infinitesimal conformal transformation on M. *q. e. d.*

In the following we write ρ instead of Ω.

Proposition 2. *The vertical components $(v^{\bar{h}})$ of X can be written as the following form:*

(2.4) $v^{\bar{h}} = y^a A^h_a + B^h$,

where A^h_a and B^h are the components of a certain (1,1) tensor field A and a certain contravariant vector field B on M, respectively.

Proof. From the equation (2.3), we can get

$$g_{mi}X_{\bar{k}}X_{\bar{j}}(v^{\bar{m}})+g_{mj}X_{\bar{k}}X_{\bar{i}}(v^{\bar{m}})=0.$$

Then we have

$$\begin{aligned}
g_{mi}X_{\bar{k}}X_{\bar{j}}(v^{\bar{m}}) &= -g_{mj}X_{\bar{k}}X_{\bar{i}}(v^{\bar{m}}) \\
&= -X_{\bar{i}}(g_{mj}X_{\bar{k}}(v^{\bar{m}}) \\
&= -X_{\bar{i}}(-g_{mk}X_{\bar{j}}(v^{\bar{m}})+2\rho g_{jk}) \\
&= g_{mk}X_{\bar{i}}X_{\bar{j}}(v^{\bar{m}}) \\
&= X_{\bar{j}}(g_{mk}X_{\bar{i}}(v^{\bar{m}}) \\
&= X_{\bar{j}}(-g_{mi}X_{\bar{k}}(v^{\bar{m}})+2\rho g_{ki}) \\
&= -g_{mi}X_{\bar{j}}X_{\bar{k}}(v^{\bar{m}}) \\
&= -g_{mi}X_{\bar{k}}X_{\bar{j}}(v^{\bar{m}}),
\end{aligned}$$

which implies $g_{mj}X_{\bar{k}}X_{\bar{i}}(v^{\bar{m}})=0$. This shows $X_{\bar{j}}(v^{\bar{m}})$ depend only on the variables (x^h). Hence $v^{\bar{h}}$ can be written as $v^{\bar{h}}=y^aA_a^h+B^h$, where A_a^h and B^h are certain function on M. The coordinate transformation rule implies A_a^h and B^h are the components of a certain (1, 1) tensor field A and a certain contravariant vector field B. *q. e. d.*

Substituting the equation (2.4) into the equation (2.2) and (2.3), then by Proposition 1, we can get

(2.5) $A_{ij}-\nabla_j v_i+\nabla_j B_i=0,$

(2.6) $\nabla_i A_j^h+K_{aij}{}^h v^a=0,$

(2.7) $A_{ij}+A_{ji}=2\rho g_{ij}.$

PROPOSITION 3. *The vector field $B=(B^h)$ is an infinitesimal isometry on M.*

PROOF. From the equation (2.5), (2.7) and Proposition 1, we have

$$\mathcal{L}_B g_{ij}=\nabla_j B_i+\nabla_i B_j=0,$$

thus the vector field B is an infinitersimal isometry on M. *q. e. d.*

PROPOSITION 4. *The scalar function ρ on M is a constant function.*

PROOF. From the equation (2.6) and (2.7), we have

$$2\nabla_k\rho g_{ij}=\nabla_k(A_{ij}+A_{ji})=-K_{akji}v^a-K_{akij}v^a=0.$$

Thus the scalar function ρ on M *is* constant. *q. e. d.*

By Proposition 4, the vector field X on $T(M)$ and the vector field V on M both become infinitesimal homothetic transformations.

Conversely let $V=(v^h)$ be an infinitesimal homothetic transformation on M that is, there exists a constant c such that $\mathcal{L}_V g_{ij}=2cg_{ij}$. Then we define the vector field X on $T(M)$ as follows

$$X=v^hX_h+y^a\nabla_a v^hX_{\bar{h}}.$$

PROPOSITION 5. *The vector field X on $T(M)$ defined above is an infinitesimal* *homothetic transformation.*

PROOF. By means of Lemma 3, we can compute $\pounds_X G_{II}$ and $\pounds_X G_{III}$.

$$\begin{aligned}
\pounds_X G_{II} &= -2g_{jm}\{y^b v^c K_{icb}{}^m - y^a \nabla_a v^b \Gamma^m_{bi} - X_i(y^a \nabla_a v^m)\} dx^j dx^i \\
&\quad + 2\{2cg_{ji} - g_{jm}\nabla_i v^m + g_{jm} X_{\bar{i}}(y^a \nabla_a v^m)\} dx^j \delta y^i \\
&= -2g_{jm} y^a \{v^c K_{ica}{}^m - \nabla_a v^b \Gamma^m_{bi} - \partial_i \nabla_a v^m + \Gamma^b_{ai} \nabla_b v^m\} dx^j dx^i + 4cg_{ji} dx^j \delta y^i \\
&= 2g_{jm} y^a \{\nabla_i \nabla_a v^m + K_{cia}{}^m v^c\} dx^j dx^i + 4cg_{ji} dx^j \delta y^i \\
&= 2g_{jm} y^a \pounds_V \Gamma^m_{ia} dx^j dx^i + 4cg_{ji} dx^j \delta y^i \\
&= 4cg_{ji} dx^j \delta y^j \\
&= 2cG_{II}.
\end{aligned}$$

$$\begin{aligned}
\pounds_X G_{III} &= -2g_{mi}\{y^b v^c K_{jcb}{}^m - y^a \nabla_a v^b \Gamma^m_{bj} - X_j(y^a \nabla_a v^m)\} dx^j \delta y^i + 2g_{mj} X_{\bar{i}}(y^a \nabla_a v^m) \delta y^j \delta y^i \\
&= 2g_{mj} \nabla_i v^m \delta y^j \delta y^i \\
&= \pounds_V g_{ji} \delta y^j \delta y^i \\
&= 2cg_{ji} \delta y^j \delta y^i \\
&= 2cG_{III}.
\end{aligned}$$

Thus we have $\pounds_X(G_{II} + G_{III}) = 2c(G_{II} + G_{III})$. This shows the vector field X on $T(M)$ is an infinitesimal homothetic transformation. *q. e. d.*

PROOF of THEOREM. Summing up Proposition 1～Proposition 5, it is clear that the correspondence $X \rightarrow V$ gives a homomorphism of the Lie algebra of infinitesimal fibre-preserving conformal transformations of $T(M)$ onto the Lie algebra of infinitesimal homothetic transformations of M, and the kernel of this homomorphism is naturally isomorphic onto the Lie algebra of infinitesimal isometries of M.

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